Matrix calculus in Control theory. Expansion and estimation for the matrix power $(A+E)^{-k}$ *

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Abstract

The power $(A+E)^{-k}$ is expanded and estimated, neglecting the terms of third and higher order in E. These results are of interest for perturbation analysis of matrix equations including matrix power of order $k \geq 2$. A numerical example demonstrates the effectiveness of the estimate proposed.

Keywords: Matrix powers, Perturbation analysis, Matrix equations, Control theory;

1 Introduction

Consider the matrix power A^{-k}, where A is an invertible complex matrix of order n and $k \geq 2$ is a positive integer. The problem of computing A^{-k} arises when solving some function of matrices and equations in Control theory, e.g. $X - A^*X^{-1}A = Q$, see [1, 2, 3, 4] and their references, $X - A^*X^{-n}A = Q$ [1, 5, 6, 7, 8, 9]. For the case A^k, which represent the transition matrix of discrete-time linear control system, explicit formulas for the coefficients in the expansion of A^k via its first $n-1$ powers as well as a computational algorithm for powers of matrices and functions of matrices are presented by Kantor and Trishin in [10]. Estimates for different norms of A^k are proposed by M. Konstantinov in Appendix A in [11] and by the author in [12].

The matrix power is in general a difficult operation from computation point of view. Moreover, the result obtained by a numerically stable method in finite precision arithmetic, is always contaminated with rounding errors. Finally, instead of the exact value of A^{-k} we obtain the solution of a slightly perturbed problem $(A+E)^{-k}$. Here the matrix E is the perturbation in the matrix A. Usually the inequality $\|E\|/\|A\| << 1$ is fulfilled reflecting the fact that the relative perturbation is relatively small. Here $\|.\|$ is the spectral $\|.\|_2$ or Frobenius $\|.\|_F$ norm in $\mathbb{C}^{n\times n}$.

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In this note the power $(A+E)^{-k}$ is expanded. A bound of the error in the result, restricted to the terms of second order in E is proposed. The bound may be used for perturbation analysis of matrix equations including matrix power of order $k \geq 2$.

2 Statement of the problem

Consider the power of a complex invertible matrix A^{-k}. The perturbed matrix is obtained by replacing a nominal value A with $A+E$

$$F(A+E) = (A+E)^{-k}. \tag{1}$$

The aim of this note is to expand the matrix power and to propose a bound for the error in the result, which is convenient for perturbation analysis.

3 Expansion of the matrix power

The expansion of the matrix power will be obtained, using the well known expression of the case $k=1$.

$$\begin{aligned} (A+E)^{-1} &= A^{-1} - A^{-1}EA^{-1} + (A^{-1}E)^2A^{-1} - (A^{-1}E)^3(I_n + A^{-1}E)^{-1}A^{-1} \\ &= A^{-1} - A^{-1}EA^{-1} + (A^{-1}E)^2A^{-1} + O(\|E\|^3). \end{aligned} \tag{2}$$

For convenience only the terms of first and second order in E will be used. The matrix power $(A+E)^{-2}$ represents the product $(A+E)^{-1}(A+E)^{-1}$. Hence for $k=2$ in view of (2) we obtain

$$\begin{aligned} (A+E)^{-2} &= (A^{-1} - A^{-1}EA^{-1} + (A^{-1}E)^2A^{-1})^{-2} \\ &= A^{-2} - A^{-2}EA^{-1} - A^{-1}EA^{-2} \\ &+ A^{-2}(EA^{-1})^2 + (A^{-1}E)^2A^{-2} + (A^{-1}EA^{-1})^2 + O(\|E\|^3). \end{aligned}$$

3.1 Case $k=3$

Although the matrix product is not commutative, we obtain one expression for $(A+E)^{-3}$:

$$\begin{aligned} (A+E)^{-3} &= (A+E)^{-2}(A+E)^{-1} = (A+E)^{-1}(A+E)^{-2} \\ &= A^{-3} - A^{-3}EA^{-1} - A^{-2}EA^{-2} - A^{-1}EA^{-3} \\ &+ A^{-3}(EA^{-1})^2 + A^{-2}EA^{-1}EA^{-2} + (A^{-1}E)^2A^{-3} \\ &+ A^{-2}EA^{-2}EA^{-1} + A^{-1}EA^{-3}EA^{-1} + A^{-1}EA^{-2}EA^{-2} + O(\|E\|^3). \end{aligned}$$

3.2 General case $k \geq 2$

In what follows for the expansion of the matrix power $(A+E)^{-k}$ for $k \geq 2$ we obtain

$$(A+E)^{-k} = A^{-k} + N(A,E),$$

where

$$N(A,E) = -\sum_{i=1}^{k} A^{-i} E A^{i-k-1} + G(A,E) + O(\|E\|^3).$$

Here $G(A,E)$ contains second order terms in E,

$$\begin{aligned} G(A,E) &= \sum_{i=1}^{k} A^{-i} E A^{-1} E A^{i-k-1} + \sum_{i=2}^{k} A^{1-i} E A^{-2} E A^{i-k-1} \\ &+ \sum_{i=3}^{k} \left(\sum_{j=1}^{k-i+1} A^{-j} E A^{-i} E A^{-k+i+j-2} \right). \end{aligned}$$

4 Estimation of $N(A,E)$

Denote by ξ the Frobenius norm of E $\xi := \|E\|_{\mathrm{F}}$, and by μ the spectral norm of A^{-1} $\mu := \|A^{-1}\|_2$.

Suppose that $\xi \leq \rho$, where ρ is a positive quantity. A bound for the error $\|N(A,E)\|_2$ is

$$\|N(A,E)\|_2 \leq e(\rho) := k\mu^{k+1}\rho + (2k-1)\mu^{k+2}\rho^2 + \left(\sum_{i=3}^{k} \sum_{j=1}^{k-i+1} \mu^{k+2}\rho^2 \right). \qquad (3)$$

As for the sum in (3) is fulfilled

$$\sum_{i=3}^{k} \sum_{j=1}^{k-i+1} \mu^{k+2}\rho^2 = \binom{k}{2} \mu^{k+2}\rho^2,$$

for the bound $e(\rho)$ we obtain

$$e(\rho) = k\mu^{k+1}\rho + \left(2k - 1 + \binom{k}{2} \right) \mu^{k+2}\rho^2. \qquad (4)$$

5 Numerical example

Consider the matrix power A^{-k} with identity matrix A of order 3 and $k = 2, 5, 8$.

The perturbation E in A is taken as

$$E = 10^{(-j)} \begin{bmatrix} 1+i & 1+i & 1+i \\ 1+i & 1+i & 1+i \\ 1+i & 1+i & 1+i \end{bmatrix}, \text{ for } j = 1, 3, 5, 7, 9.$$

The spectral norm of the error in the result $(A+E)^{-k}$ is estimated by the bound $e(\rho)$ from (4).

The results obtained for $j = 1, 3, 5, 7, 9$ and for $k = 2, 5, 8$ are shown at the tables below.

In all cases the estimated value is near to the quantity, predicted by the bound.

6 Concluding notes

The power $(A+E)^{-k}$, when $k \geq 2$, is expanded. A bound for the error in the result is proposed. A numerical example illustrates the effectiveness of the bound proposed.

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Table 1. $k = 2$

j	1	3	5	7	9
$\|N(A,E)\|_2$	4.08×10^{-1}	5.97×10^{-3}	6.00×10^{-5}	6.00×10^{-7}	6.00×10^{-9}
$e(\rho)$	9.60×10^{-1}	6.04×10^{-3}	6.00×10^{-5}	6.00×10^{-7}	6.00×10^{-9}

Table 2. $k = 5$

j	1	3	5	7	9
$\|N(A,E)\|_2$	7.31×10^{-1}	1.49×10^{-2}	1.50×10^{-4}	1.50×10^{-6}	1.50×10^{-8}
$e(\rho)$	3.21	1.52×10^{-2}	1.50×10^{-4}	1.50×10^{-6}	1.50×10^{-8}

Table 3. $k = 8$

j	1	3	5	7	9
$\|N(A,E)\|_2$	8.77×10^{-1}	2.37×10^{-2}	2.40×10^{-4}	2.40×10^{-6}	2.40×10^{-8}
$e(\rho)$	6.27	2.44×10^{-2}	2.40×10^{-4}	2.40×10^{-6}	2.40×10^{-8}