COMPARISON OF PERTURBATION BOUNDS FOR THE MATRIX EQUATION $X = A_1 + A_2^{\mathrm{H}} X^{-1} A_2$

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Abstract

The paper deals with the perturbation estimates proposed by Xu [4], Sun, Xu [3] and Konstantinov *et al* [2] for evaluating the sensitivity of the solution to the complex fractional-affine matrix equation $X = A_1 + A_2^{\mathrm{H}} X^{-1} A_2$ relative to rounding and parameter errors. The effectiveness and reliability of the different methods are analyzed by experiments with numerical examples.

Key Words: Perturbations, complex fractional-affine matrix equation, sensitivity.
MSC 2000: 15A24.

1 Introduction

Shu-Fang Xu in [4], Ji-gunag Sun in [3] and Konstantinov *et al* in [2] consider the sensitivity of the complex fractional-affine matrix equation

$$X = A_1 + A_2^{\mathrm{H}} X^{-1} A_2, \tag{1}$$

where $A_1 \in \mathbb{C}^{n\times n}$ and the solution $X \in \mathbb{C}^{n\times n}$ are Hermitian positive definite matrices. Different schemes for estimating the sensitivity of the solution relative

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to rounding and parameter errors are presented in the above papers. Therefore, it is interesting to compare the effectivness and the field of application of these different perturbation results.

Perturbation analysis of real equations of type (1) is done in [1].

In this paper, by means of numerical experiments, we present a comparison analysis of the effectiveness and applicability of the perturbation bounds, proposed in [4, 3, 2].

Throughout the paper we use the following notation:

$\mathbb{C}^{n\times n}$ – the space of $n \times n$ complex matrices;

$\mathbb{R}_+ = [0, \infty)$;

$\mathbb{H}^{n\times n}$ – the space of Hermitian matrices;

$A^\top$ – the transpose of the matrix A;

$\bar{A}$ – the complex conjugate of A;

$A^{\mathrm{H}} = \bar{A}^\top$;

$\mathrm{vec}(A) \in \mathbb{C}^{n^2}$ – the column-wise vector representation of the matrix $A \in \mathbb{C}^{n\times n}$;

$\mathrm{Mat}(\mathbf{L}) \in \mathbb{C}^{n^2\times n^2}$ – the matrix representation of the linear matrix operator $\mathbf{L} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$;

I_n – the identity $n \times n$ matrix;

$A \otimes B = [a_{pq}B]$ – the Kronecker product of the matrices $A = [a_{pq}]$ and B;

$\|\cdot\|_2$ – the Euclidean norm in $\mathbb{C}^n$ or the spectral (or 2-) norm in $\mathbb{C}^{n\times n}$;

$\|\cdot\|_{\mathrm{F}}$ – the Frobenius (or F-) norm in $\mathbb{C}^{n\times n}$;

$\|\cdot\|$ – a replacement of either $\|\cdot\|_2$ or $\|\cdot\|_{\mathrm{F}}$;

$z^{\mathbb{R}} \in \mathbb{R}^{2n}$ – the real version of the vector $z \in \mathbb{C}^n$;

$\Gamma^{\mathbb{R}} \in \mathbb{R}^{2n\times 2n}$ – the real version of the matrix $\Gamma \in \mathbb{C}^{n\times n}$;

$\Theta(\Gamma, \Delta)$ – the matrix of the real version of the operator $z \to \Gamma z + \Delta\bar{z}$;

$\Pi_{n^2} \in \mathbb{C}^{n^2\times n^2}$ – the vec-permutation matrix such that $\mathrm{vec}(Y^\top) = \Pi_{n^2}\mathrm{vec}(Y)$ for each $Y \in \mathbb{C}^{n\times n}$.

The notation ':=' stands for 'equal by definition'. The sub-indices k, l take values 1,2.

2 Statement of the problem

Consider the matrix equation (1). The round–off and parameter errors, accompanying the numerical solution of the equation, are represented by equivalent perturbations in the matrix coefficients, $A_k \to \tilde{A}_k := A_k + \delta A_k$. They lead to a perturbation in the solution $X \to \tilde{X} = X + \delta X$, where $\delta A_1, \delta X \in \mathbb{H}^{n\times n}$ and $\delta A_2 \in \mathbb{C}^{n\times n}$ and for some $\delta_k \geq 0$ we have $\|\delta A_k\|_{\mathrm{F}} \leq \delta_k$.

The perturbed equation (1) is

$$\tilde{X} = \tilde{A}_1 + \tilde{A}_2^{\mathrm{H}}\tilde{X}^{-1}\tilde{A}_2. \qquad (2)$$

The purpose of the norm-wise perturbation analysis of equation (1) is to derive bounds for $\delta_X := \|\delta X\|_{\mathrm{F}}$ as a linear or non-linear function of the perturbations $\|\delta A_k\|_{\mathrm{F}}$ in the data.

3 Sensitivity estimates

3.1 The estimate of Xu [4]

In [4] Xu proves the following theorem.

Theorem 1. *Let $A_2, \tilde{A}_2, A_1, \tilde{A}_1 \in \mathbb{C}^{n\times n}$ with A_1 and $\tilde{A}_1$ Hermitian positive definite. If*

$$\begin{aligned} &\|A_2\|_2\|A_1^{-1}\|_2 < \frac{1}{2}, \\ &\|\tilde{A}_2 - A_2\|_2 < \frac{1}{2}\left(\frac{1}{2} - \|A_2\|_2\|A_1^{-1}\|_2\right)\|A_1^{-1}\|_2^{-1}, \\ &\|\tilde{A}_1 - A_1\|_2 \le \left(\frac{1}{2} - \|A_2\|_2\|A_1^{-1}\|_2\right)\|A_1^{-1}\|_2^{-1}, \end{aligned} \tag{3}$$

then the solutions X and $\tilde{X}$ of the matrix equations (1) and (2) exist and satisfy that

$$\frac{\|\tilde{X} - X\|_2}{\|X\|_2} \le \zeta_1 := \frac{1}{\frac{1}{2} - \|A_2\|_2\|A_1^{-1}\|_2}\left(\frac{\|\tilde{A}_2 - A_2\|_2}{\|A_2\|_2} + \frac{\|\tilde{A}_1 - A_1\|_2}{\|A_1\|_2}\right). \tag{4}$$

3.2 The estimate of Sun and Xu [3]

Denote

$$B := X^{-1}A_2,\ \alpha := \|A_2\|_2,\ \beta := \|B\|_2,\ \zeta := \|X^{-1}\|_2.$$

Let, in the real case,

$$\begin{aligned} L &:= I - B^\top \otimes B^{\rm H} = I - \left(X^{-1}A_2\right)^\top \otimes \left(X^{-1}A_2\right)^{\rm H}, \\ Q &:= L^{-1}\left(I \otimes B^{\rm H} + (B^\top \otimes I_n)\Pi_{n^2}\right) \\ &= L^{-1}(I_n \otimes (X^{-1}A)^{\rm H}) + L^{-1}\left((X^{-1}A_2)^\top \otimes I_n\right)\Pi_{n^2} \end{aligned}$$

and $q := \|Q\|_{\rm F}$, $l := \|L^{-1}\|_{\rm F}^{-1}$. Similarly let, in the complex case,

$$\begin{aligned} L^{-1} &=: S + i\Sigma, \\ L^{-1}\left(I_n \otimes B^{\rm H}\right) &= =: U_1 + i\Omega_1, \\ L^{-1}\left(B^\top \otimes I_n\right)\Pi_{n^2} &= =: U_2 + i\Omega_2, \\ S_c &:= \begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix}, \\ U_c &:= -\begin{bmatrix} U_1 + U_2 & \Omega_2 - \Omega_1 \\ \Omega_1 + \Omega_2 & U_1 - U_2 \end{bmatrix} \end{aligned}$$

and $q := \|U_c\|_{\rm F}$, $l := \|S_c\|_{\rm F}^{-1}$. Denote

$$\begin{aligned} \epsilon &= \frac{1}{l}\|\delta A_1\|_{\rm F} + q\|\delta A_2\|_{\rm F} + \frac{\zeta}{l}\|\delta A_2\|_{\rm F}^2, \\ \delta &= \frac{\zeta}{l}\left((\alpha + \|\delta A_2\|_{\rm F})\zeta + \beta\right)\|\delta A_2\|_{\rm F}. \end{aligned}$$

In [3] Sun and Xu prove the following theorem.

Theorem 2. *If*

$$\delta < \min\left\{1, \frac{(1-\beta)(\alpha\zeta + \beta)}{l}\right\}$$

and

$$\epsilon < \min\left\{\frac{l(1-\delta)^2}{\zeta(l+2\beta^2+l\delta+2\sqrt{(l\delta+\beta^2)(l+\beta^2)})},\right.$$
$$\left.\frac{(l-\delta)((1-\beta)(\alpha\zeta+\beta)-l\delta)}{\zeta((1+\beta)(\alpha\zeta+\beta)+l\delta)}\right\},$$

then the perturbed matrix equation (2) has a solution $\tilde{X}$, and moreover,

$$\|\tilde{X}-X\|_{\mathrm{F}} \le \zeta_2 := \frac{2l\epsilon}{l(1+\zeta\epsilon-\delta)+\sqrt{l^2(1+\zeta\epsilon-\delta^2)^2-4\zeta l\epsilon(l+\beta^2)}}. \quad (5)$$

3.3 The estimate of Konstantinov *et al* [2]

Denote $\xi := \|\delta X\|_{\mathrm{F}}$ and $\delta := [\delta_1,\delta_2]^\top = \left[\|\delta A_1\|_{\mathrm{F}}, \|\delta A_2\|_{\mathrm{F}}\right]^\top \in \mathbb{R}^2_+$.

In [2] the following local bound is derived for the perturbation δX in the solution X of equation (1)

$$\xi \le g(\delta) + O(\|\delta\|_2),\ \delta \to 0, \quad (6)$$

where

$$\begin{aligned}
g(\delta) &:= \min\left\{\|[M_1^0, M_2^0]\|_2\|\delta\|_2, \sqrt{\delta^\top M^0\delta}\right\}, \quad (7)\\
L &:= I_{n^2} - (X^{-1}A_2)^\top \otimes (A_2^{\mathrm{H}}X^{-1}),\\
M_1 &:= -L^{-1} =: \Gamma + i\Delta,\\
M_{21} &:= +L^{-1}\left(I_n \otimes (A_2^{\mathrm{H}}X^{-1})\right) =: \Gamma_{21} + i\Delta_{21},\\
M_{22} &:= +L^{-1}\left(((X^{-1}A_2)^\top \otimes I_n)\Pi_{n^2}\right) =: \Gamma_{22} + i\Delta_{22}\\
M_1^0 &:= \begin{bmatrix} \Gamma & -\Delta \\ \Delta & \Gamma \end{bmatrix},\quad M_2^0 := \begin{bmatrix} \Gamma_{21}+\Gamma_{22} & \Delta_{22}-\Delta_{21} \\ \Delta_{21}+\Delta_{22} & \Gamma_{21}-\Gamma_{22} \end{bmatrix},
\end{aligned}$$

and $M^0 = [m^0_{kl}] \in \mathbb{R}^{2\times 2}_+$ is a symmetric matrix with elements

$$m^0_{kl} = \left\|{M_k^0}^\top M_l^0\right\|_2.$$

Let $\mu := \|X^{-1}\|_2$ and

$$\begin{aligned}
a_0(\delta) &:= g(\delta) + c_1\mu\delta_2^2,\\
a_1(\delta) &:= a_{11}\delta_2 + a_{12}\delta_2^2,\\
a_2(\delta) &:= a_{20} + a_{21}\delta_2 + a_{22}\delta_2^2,
\end{aligned}$$

where

$$\begin{aligned}
a_{11} &:= \mu\left(\left\|L^{-1}\left(I_n \otimes \left(A_2^{\mathrm{H}}X^{-1}\right)\right)\right\|_2 + \left\|L^{-1}\left(\left(X^{-1}A_2\right)^\top \otimes I_n\right)P_{n^2}\right\|_2\right),\\
a_{12} &:= c_1\mu^2,\\
a_{20} &:= \mu^3\left\|L^{-1}\left(A_2^\top \otimes A_2^{\mathrm{H}}\right)\right\|_2,\\
a_{21} &:= \mu^3\left\|L^{-1}\left(A_2^\top \otimes I_n\right)\Pi_{n^2} + L^{-1}\left(I_n \otimes A_2^{\mathrm{H}}\right)\right\|_2,\\
a_{22} &:= c_1\mu^3.
\end{aligned}$$

The following theorem is proved in [2]
Theorem 3. *Let*

$$\delta \in \Omega := \left\{ \delta \in \mathbb{R}_+^2 \ : \ a_1 - \mu a_0 + 2\sqrt{a_0(a_2 + \mu(1-a_1))} \le 1 \right\}. \qquad (8)$$

Then the non-local perturbation bound

$$\begin{aligned} \|\delta X\|_{\rm F} \le f(\delta) &:= \frac{2a_0(\delta)}{1 - a_1 + \mu a_0 + \sqrt{d(\delta)}}, \qquad (9) \\ d(\delta) &:= (1 - a_1(\delta) + \mu a_0(\delta))^2 - 4a_0(\delta)(a_2(\delta) + \mu(1 - a_1(\delta))) \end{aligned}$$

is valid for equation (1).

4 Experimental results

Example 1. The model from Example 5.1 in [3] is used. Consider equation (1) with matrix coefficients $A_1 = I_2$, $A_2 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, where $a = 0.5 - 10^{-k}$. The solution of the equation is $X = \mathrm{diag}(1, 1-a^2)$. The perturbations in the data are taken as:

$$\delta A_1 = \begin{bmatrix} -0.4326 & -0.7701 \\ 0.7701 & 0.2877 \end{bmatrix} \times 10^{-8}, \quad \delta A_2 = \begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix} \times 10^{-9},$$

for $k = 1, 3, 5, 7, 9$.

In Table 1 we give the numerical results for the relative perturbations bounds ζ_1 (4) of Xu [4]; $\zeta_2/\|X\|_2$ (5) of Sun and Xu; the local bound $g(\delta)/\|X\|_2$ (6) and the non-local bound $f(\delta)/\|X\|_2$ (9) of Konstantinov *et al* [2].

Table 1.

k	1	3	5	7	9
ζ_1	1.44×10^{-7}	1.38×10^{-5}	1.38×10^{-3}	1.38×10^{-1}	$1.38\times10^{+1}$
$\zeta_2/\|X\|_2$	3.39×10^{-8}	3.46×10^{-8}	3.46×10^{-8}	3.46×10^{-8}	3.46×10^{-8}
$g(\delta)/\|X\|_2$	1.40×10^{-8}	1.48×10^{-8}	1.48×10^{-8}	1.48×10^{-8}	1.48×10^{-8}
$f(\delta)/\|X\|_2$	1.40×10^{-8}	1.48×10^{-8}	1.48×10^{-8}	1.48×10^{-8}	1.48×10^{-8}

Let now $a = 0.99$ and the perturbations in the matrix coefficients be

$$\delta A_1 = \begin{bmatrix} -0.4326 & -0.7701 \\ 0.7701 & 0.2877 \end{bmatrix} \times 10^{-k}, \quad \delta A_2 = \begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix} \times 10^{-k}$$

for $k = 6, \ldots, 10$.

For this example the estimate ζ_1 from Xu [4] is not valid, since the conditions (3) from Theorem 1 are not satisfied. The results for ζ_2 (5) of Sun and Xu, the local bound $g(\delta)$ (6) and the non-local bound $f(\delta)$ (9) of Konstantinov *et al.* are listed in Table 2.

The case when the non-local estimate $f(\delta)$ is not valid, since the existence condition $\delta \in \Omega$ is violated, is denoted by asterisk.

Table 2.

k	6	7	8	9	10
ζ_2	5.47×10^{-6}	5.43×10^{-7}	5.43×10^{-8}	5.43×10^{-9}	5.43×10^{-10}
$g(\delta)$	4.27×10^{-6}	4.27×10^{-7}	4.27×10^{-8}	4.27×10^{-9}	4.27×10^{-10}
$f(\delta)$	*	4.52×10^{-7}	4.29×10^{-8}	4.27×10^{-9}	4.27×10^{-10}

Example 2. For this example we use the model from [2]. Consider the complex fractional-affine matrix equation (1) with matrices

$$\begin{aligned} A_2 &= \begin{bmatrix} 0.2190+0.0535i & 0.6793+0.0077i & 0.5194+0.4175i \\ 0.0470+0.5297i & 0.9347+0.3834i & 0.8310+0.6868i \\ 0.6789+0.6711i & 0.3835+0.0668i & 0.0346+0.5890i \end{bmatrix}, \\ A_1 &= I_3 + A_2^{\mathrm{H}} A_2. \end{aligned}$$

The perturbations in the data are taken as

$$\begin{aligned} \delta A_2 &= 10^{(-k)} \begin{bmatrix} 1+i & 1+i & 1+i \\ 1+i & 1+i & 1+i \\ 1+i & 1+i & 1+i \end{bmatrix}, \\ \delta A_1 &= \delta A_2 + (A_2+\delta A_2)^{\mathrm{H}}(I_3+\delta A_2)^{-1}(A_2+\delta A_2) - A_2^{\mathrm{H}} A_2 \end{aligned}$$

for $k = 10, 9, ..., 2$.

This problem was designed so as to have solutions $X = I_3$ and $X + \delta X = I_3 + \delta A_2$ of the unperturbed and perturbed equations respectively.

The perturbation $\|\delta X\|_F$ in the solution is estimated by the bound ζ_2 (5) of Sun and Xu [3], the local bound $g(\delta)$ (6) and the non-local bound $\rho(\delta)$ (9) of Konstantinov *et al* [2]. For this example the bound ζ_1 (see (4)) of Xu [4] is not valid, since the conditions (3) are violeted.

The results obtained for different values of k are shown at Table 3. When k decreases from 10 to 2 the non-local estimates ζ_2 (5) and $\rho(\delta)$ (9) are slightly more pessimistic than the local bound $g(\delta)$. The cases when the non-local estimates are not valid, since the existence conditions are violated, are denoted by asterisk.

Table 3.

k	$\|\delta X\|_F$	ζ_2	$g(\delta)$	$\rho(\delta)$
10	4.24×10^{-10}	4.40×10^{-9}	3.40×10^{-9}	3.40×10^{-9}
9	4.24×10^{-9}	4.40×10^{-8}	3.40×10^{-8}	3.40×10^{-8}
8	4.24×10^{-8}	4.40×10^{-7}	3.40×10^{-7}	3.40×10^{-7}
7	4.24×10^{-7}	4.40×10^{-6}	3.40×10^{-6}	3.40×10^{-6}
6	4.24×10^{-6}	4.40×10^{-5}	3.40×10^{-5}	3.40×10^{-5}
5	4.24×10^{-5}	4.42×10^{-4}	3.40×10^{-4}	3.40×10^{-4}
4	4.24×10^{-4}	4.66×10^{-3}	3.40×10^{-3}	3.43×10^{-3}
3	4.24×10^{-3}	*	3.40×10^{-2}	3.71×10^{-2}
2	4.24×10^{-2}	*	3.35×10^{-1}	*

5 Concluding remarks

The results of the experimental analysis show that in the given cases the non–linear method of Konstantinov *et al.* [2] for estimating the sensitivity of the solution to equation(1) is superior to the methods of Xu [4] and Sun and Xu [3] with respect of closeness to the estimated quantity.

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СРАВНЕНИЕ НА ПЕРТУРБАЦИОННИТЕ ГРАНИЦИ ЗА МАТРИЧНОТО УРАВНЕНИЕ $X = A_1 + A_2^{\mathrm{H}} X^{-1} A_2$

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Резюме

Сравнени са резултатите на Ксю [4], Сун и Ксю [3] и Константинов и др. [2] за оценяване на чуствителността на решението на комплексното матрично дробно-линейно уравнение $X = A_1 + A_2^{\mathrm{H}} X^{-1} A_2$ относно смущения в матричните коефициенти. Надеждността и ефективността на различните оценки са анализирани с помощта на числени експерименти.

СРАВНЕНИЕ ПЕРТУРБАЦИОННЫХ ГРАНИЦ ДЛЯ МАТРИЧНОГО УРАВНЕНИЯ $X = A_1 + A_2^{\mathrm{H}} X^{-1} A_2$

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Резюме

Сделано сравнение результатов Ксю [4], Сун и Ксю [3] и Константинов и др. [2] для оценки чуствительности решения комплексного матричного дробно-линейного уравнения $X = A_1 + A_2^{\mathrm{H}} X^{-1} A_2$ относительно смущений в матричных коефициентах. Надеждность и еффективность различных оценок анализированны численными экспериментами.