Perturbation analysis for the matrix equation $\mathbf{X} = \mathbf{A_1} + \sigma \mathbf{A_2^H X^{-2} A_2}$, $\sigma = \pm 1$

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Abstract

Condition numbers, local and non-local perturbation bounds are derived for the complex matrix equation

The technique used is based on Lyapunov majorants and fixed point principles. An illustrative numerical example is given.

Key Words: Perturbation analysis, Matrix equations;
MSC 2000: 15A24.

1 Introduction and notation

In this paper a complete perturbation analysis of the nonlinear complex matrix equation

$$F(X, A) := X - A_1 - \sigma A_2^{\mathrm{H}} X^{-2} A_2 = 0, \quad \sigma = \pm 1 \tag{1}$$

with data matrices $A_1, A_2 \in \mathbb{C}^{n \times n}$, solution $X \in \mathbb{C}^{n \times n}$ and A_1 - positive definite is presented. Equation of this type with A_1 the identity matrix is studied by several authors. Sufficient conditions for existence of positive definite solution are obtained in [3, 4, 7]. Necessary and sufficient conditions for existence of Hermitian positive definite solution in case $\sigma = +1$ and A_2 normal are given in [18]. Iterative methods for obtaining positive definite solution are constructed [3, 4, 7, 18]. For the application areas in which the equation arises, see the references given in [4, 5, 7].

The more general non-linear matrix equations $X + A^* \mathcal{F}(X) A = Q$ [1, 16] and $X + A^* X^{-n} A = Q$, $X + A^* X^{-n} A = I$ [3, 5, 6, 10, 17] are investigated. The Hermitian positive definite solutions of the equations and its properties are studied. Theorems of necessary and sufficient conditions for the existence of a solution are proved. Perturbation upper bounds for the solution are derived. Iterative methods for computing a positive definite solution are proposed.

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The real matrix equation $X^s \pm A^\top X^{-t} A = I_n$ is considered in [14]. The properties of the solution are discussed. Sensitivity analysis is made. Iterative methods for computing the positive definite solutions are proposed.

Throughout the paper the following notation is used: $\mathbb{F}^{n\times n}$ – the space of $n \times n$ real $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ complex matrices; $\mathbb{R}_+ = [0, \infty)$; $A^\top$ – the transpose of the matrix A; $\bar{A}$ - the complex conjugate of A; $A^{\rm H} = \bar{A}^\top$; $\mathrm{vec}(A) \in \mathbb{C}^{n^2}$ – the column-wise vector representation of the matrix $A \in \mathbb{C}^{n\times n}$; $\mathrm{Mat}(\mathbf{L}) \in \mathbb{C}^{n^2\times n^2}$ – the matrix representation of the linear matrix operator $\mathbf{L} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$; I_n – the identity $n \times n$ matrix; $A \otimes B = [a_{pq}B]$ – the Kronecker product of the matrices $A = [a_{pq}]$ and B; $\|\cdot\|_2$ – the Euclidean norm in $\mathbb{F}^n$ or the spectral (or 2-) norm in $\mathbb{F}^{n\times n}$; $\|\cdot\|_{\rm F}$ – the Frobenius (or F-) norm in $\mathbb{C}^{n\times n}$; $\|\cdot\|$ – a replacement of either $\|\cdot\|_2$ or $\|\cdot\|_{\rm F}$; $z^{\mathcal{R}} \in \mathbb{R}^{2n}$ - the real version of $z \in \mathbb{C}^n$; $\Gamma^{\mathbb{R}} \in \mathbb{R}^{2n\times 2n}$ - the real version of $\Gamma \in \mathbb{C}^{n\times n}$; $\Theta(\Gamma, \Delta)$ - the matrix of the real version of the operator $z \to \Gamma z + \Delta \bar{z}$.

The notation ':=' stands for 'equal by definition'.

In the following the subindexes k, l take values 1,2.

The paper is organised as follow. In Section 2 a local perturbation analysis is presented. The perturbations $\delta A := (\delta A_1, \delta A_2)$, δX in the data matrix collection $A = (A_1, A_2)$ and the solution X are estimated in terms of the Frobenius matrix norm $\|\cdot\|_{\rm F}$. Explicit expression for the individual condition number of X relative to perturbation in A is obtained. Rewriting (1) as an equivalent matrix equation for the perturbation in the solution, non-linear non-local bound is obtained in Section 3. The technique used is based on Lyapunov majorants and fixed point principles [9]. In Section 4 a numerical example demonstrate the effectiveness of the bounds proposed.

2 Local perturbation analysis

Let A_i and X be slightly perturbed to $A_i + \delta A_i$, $X + \delta X$, where $\delta A_i, \delta X \in \mathbb{C}^{n\times n}$. Let the numbers $\delta_k \geq 0$ be given and suppose that $\alpha_k := \|\delta A_k\|_{\rm F} \leq \delta_k$. The perturbed equation is obtained from(1) by replacing the nominal value A with $A + \delta A$ and X with $X + \delta X$

$$F(X + \delta X, A + \delta A) = 0. \tag{2}$$

For δ_k sufficiently small equation (2) has a solution $X + \delta X$, depending on δA.

Rewrite (2) as an equivalent equation for the perturbation δX in X

$$\begin{aligned} F(X + \delta X, A + \delta A) &= F(X, A) + F_X(X, A)(\delta X) + F_{A_1}(X, A)(\delta A_1) \\ &+ F_{A_2}(X, A)(\delta A_2) + F_{\bar{A}_2}(X, A)(\delta \bar{A}_2) + G(X, A)(\delta X, \delta A), \end{aligned}$$

where $F_X(X, A) : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is the partial Fréchet derivative of F in X calculated at the point (X, A), and $F_{A_1}(X, A) : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is the partial Fréchet derivative of F in A_1 calculated at the point (X, A). Similarly,

$$F_{A_2}(X, A)(\delta A_2) + F_{\bar{A}_2}(X, A)(\delta \bar{A}_2),$$

is an additive but not homogeneous operator related to the partial Fréchet derivative of $F(X + \delta X, A + \delta A)$ in A_2. This is due to the fact that complex conjugation (and hence the map $A \to A^{\rm H}$) is not a linear operation.

Set

$$\mathcal{L} := F_X,\ \mathcal{L}_1 := F_{A_1},\ \mathcal{L}_2 = \mathcal{L}_{21} + \mathcal{L}_{22} := F_{A_2} + F_{\bar{A}_2},$$

then

$$\begin{aligned} F(X+\delta X, A+\delta A) &= F(X,A) + \mathcal{L}(\delta X) + \mathcal{L}_1(\delta A_1) \\ &\quad + \mathcal{L}_{21}(\delta A_2) + \mathcal{L}_{22}(\delta \bar{A}_2) + G(\delta X, \delta A). \end{aligned} \tag{3}$$

Denote $\xi := \|\delta X\|_{\mathrm{F}}$. The term $G(\delta X, \delta A)$ contains second and higher order terms in $\delta X, \delta A$,

$$G(\delta X, \delta A) = O(\xi^2 + \alpha_1^2 + \alpha_2^2),\ \xi + \alpha_1 + \alpha_2 \to 0.$$

The partial Fréchet derivative of F in X is

$$\mathcal{L}(Y) = Y + \sigma A_2^{\mathrm{H}} X^{-2} Y X^{-1} A_2 + \sigma A_2^{\mathrm{H}} X^{-1} Y X^{-2} A_2,$$

and its matrix is

$$L = I_{n^2} + \sigma (X^{-1}A_2)^\top \otimes (A_2^{\mathrm{H}} X^{-2}) + \sigma (X^{-2}A_2)^\top \otimes (A_2^{\mathrm{H}} X^{-1}). \tag{4}$$

The eigenvalues of $\mathcal{L}$ are the eigenvalue of its matrix L and are equal to

$$1 + \sigma \lambda_i \left((X^{-1}A_2)^\top \otimes (A_2^{\mathrm{H}} X^{-2}) + (X^{-2}A_2)^\top \otimes (A_2^{\mathrm{H}} X^{-1}) \right), \quad i = 1, 2, \ldots, n^2.$$

Here $\lambda_i(Z)$ is the eigenvalue of the matrix Z.

The operator $\mathcal{L}$ and its matrix L are invertible iff

$$\sigma \lambda_i \left((X^{-1}A_2)^\top \otimes (A_2^{\mathrm{H}} X^{-2}) + (X^{-2}A_2)^\top \otimes (A_2^{\mathrm{H}} X^{-1}) \right) \neq -1. \tag{5}$$

In what follows we assume that the inequalities (5) hold true.

Having in mind that $F(X, A) = 0$, $F(X + \delta X, A + \delta A) = 0$ and supposing that the operator $\mathcal{L}$ is invertible we obtain from (3)

$$\delta X = -\mathcal{L}^{-1} \circ \mathcal{L}_1(\delta A) - \mathcal{L}^{-1} \circ \mathcal{L}_{21}(\delta A_2) - \mathcal{L}^{-1} \circ \mathcal{L}_{22}(\delta \bar{A}_2) + O(\|a\|^2),\ a \to 0$$

and

$$x = M_1 a_1 + M_{21} a_2 + M_{22} \bar{a}_2 + O(\|a\|^2),\ a \to 0. \tag{6}$$

Here $x := \mathrm{vec}(\delta X)$, $a_k := \mathrm{vec}(\delta A_k)$ are n^2-vectors, $a := \mathrm{vec}(\delta A) = [a_1^\top, a_2^\top]^\top \in \mathbb{C}^{2n^2}$, $M_1 = -L^{-1} L_1 \in \mathbb{C}^{n^2 \times n^2}$, is the matrix of the operator $-\mathcal{L}^{-1} \circ \mathcal{L}_1$, $M_{2k} = -L^{-1} L_{2k} \in \mathbb{C}^{n^2 \times n^2}$ is the matrix of the operator $-\mathcal{L}^{-1} \circ \mathcal{L}_{2k}$. The matrices of the operators

$$\mathcal{L}_1(Y) = -Y,\ \mathcal{L}_{21}(Y) = -\sigma A_2^{\mathrm{H}} X^{-2} Y,\ \mathcal{L}_{22} = -\sigma Y^{\mathrm{H}} X^{-2} A_2$$

are

$$\begin{aligned} L_1 &= -I_{n^2} \in \mathbb{C}^{n^2 \times n^2}, \\ L_{21} &= -\sigma I_n \otimes (A_2^{\mathrm{H}} X^{-2}) \in \mathbb{C}^{n^2 \times n^2}, \\ L_{22} &= -\sigma ((X^{-2}A_2)^\top \otimes I_n) P_{n^2} \in \mathbb{C}^{n^2 \times n^2}, \end{aligned}$$

where $P_{n^2} \in \mathbb{C}^{n^2 \times n^2}$ is the so called vec-permutation matrix such that $\mathrm{vec}(Y^\top) = P_{n^2} \mathrm{vec}(Y)$ for each $Y \in \mathbb{C}^{n \times n}$.

For the real version of x is fulfilled [12, 9]

$$x^{\mathcal{R}} = M_1^{\mathcal{R}} a_1^{\mathcal{R}} + \Theta(M_{21}, M_{22}) a_2^{\mathcal{R}} + O(\|a^{\mathcal{R}}\|^2),\ a^{\mathcal{R}} \to 0,$$

where for the complex $n \times n$ matrices $M_{21} = M_{21r} + iM_{21i}$, $M_{22} = M_{22r} + iM_{22i}$ (with M_{21r}, M_{21i}, M_{22r}, M_{22i} real)

$$\Theta(M_{21}, M_{22}) := \begin{bmatrix} M_{21r} + M_{22r} & M_{22i} - M_{21i} \\ M_{21i} + M_{22i} & M_{21r} - M_{22r} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}. \tag{7}$$

and for the product $M_1 a_1$ of a complex matrix $M_1 = M_{1r} + iM_{1i} \in \mathbb{C}^{n \times n}$ and a complex vector $a_1 = a_{1r} + ia_{1i} \in \mathbb{C}^n$ (with M_{1r}, M_{1i} and a_{1r}, a_{1i} real) we have the real version

$$(M_1 a_1)^{\mathcal{R}} := M_1^{\mathcal{R}} a_1^{\mathcal{R}} \in \mathbb{R}^{2n}$$

with

$$M_1^{\mathcal{R}} := \begin{bmatrix} M_{1r} & -M_{1i} \\ M_{1i} & M_{1r} \end{bmatrix},\quad a_1^{\mathcal{R}} := \begin{bmatrix} a_{1r} \\ a_{1i} \end{bmatrix}.$$

For the matrices M_1, M_{2k} we obtain

$$\begin{aligned} M_1 &= L^{-1}, \\ M_{21} &= \sigma L^{-1}(I_n \otimes (A_2^{\mathrm{H}} X^{-2})), \\ M_{22} &= \sigma L^{-1}(((X^{-2} A_2)^\top \otimes I_n) P_{n^2}). \end{aligned} \tag{8}$$

Recalling that $\xi := \|\delta X\|_{\mathrm{F}} = \|\mathrm{vec}(\delta X)\|_2 = \|x\|_2$ and since $\|x\|_2 = \|x^{\mathcal{R}}\|_2$ we have

$$\xi \le c\delta + O(\|\delta\|^2) = c_1\delta_1 + c_2\delta_2 + O(\|\delta\|^2),\ \delta \to 0,$$

where $c := [c_1, c_2] \in \mathbb{R}^{1\times 2}$ and $c_1 = \|M_1^0\|_2$, $M_1^0 := M_1^{\mathcal{R}}$, $c_2 = \|M_2^0\|_2$, $M_2^0 := \Theta(M_{21}, M_{22})$ and $\delta := [\delta_1, \delta_2]^\top = [\|\delta A_1\|_{\mathrm{F}}, \|\delta A_2\|_{\mathrm{F}}]^\top \in \mathbb{R}_+^2$.

Hence the absolute individual condition numbers are calculated from

$$c_1 = \left\|M_1^0\right\|_2,\ c_2 = \left\|M_2^0\right\|_2,$$

where the matrix M_1^0, is the real version of the matrix M_1, given by (4) and (8), and M_2^0 is the matrix $\Theta(M_{21}, M_{22})$, given by (7). The relative individual condition numbers are then computed from $\gamma_k = c_k \|A_k\|_{\mathrm{F}} / \|X\|_{\mathrm{F}}$.

Relation (6) also gives

$$\xi \le \mathrm{est}_2(\delta) + O(\|\delta\|^2) := \|\left[M_1^0, M_2^0\right]\|_2 \|\delta\|_2 + O(\|\delta\|^2),\ \delta \to 0,$$

and

$$\xi \le \mathrm{est}_3(\delta) + O(\|\delta\|_2) := \sqrt{\delta^\top M^0 \delta} + O(\|\delta\|^2),\ \delta \to 0,$$

where $[M_1^0, M_2^0] \in \mathbb{R}^{n^2 \times 2n^2}$ and $M^0 = [m_{kl}^0] \in \mathbb{R}_+^{2\times 2}$ is a symmetric matrix with elements $m_{kl}^0 = \left\|{M_k^0}^\top M_l^0\right\|_2$.

Since $c\delta \le \sqrt{\delta^\top M^0 \delta}$ (see [13]) we find the local perturbation estimate

$$\xi \le g(\delta) + O(\|\delta\|^2),\ \delta \to 0, \tag{9}$$

where

$$g(\delta) := \min\{\mathrm{est}_2(\delta), \mathrm{est}_3(\delta)\}. \tag{10}$$

The estimate (9), (10) allows to define the overall relative condition number as follows. Let $\delta_k = \varepsilon\|A_k\|_{\mathrm{F}}$, where $\varepsilon > 0$ (in floating point arithmetic the quantity ε may be taken as a multiple of the rounding unit). Then $g(\delta) = \varepsilon g(a^0)$, where $a^0 := [\|A_1\|_{\mathrm{F}}, \|A_2\|_{\mathrm{F}}]^\top$. Hence the relative perturbation in the solution can be estimated as

$$\frac{\|\delta X\|_{\mathrm{F}}}{\|X\|_{\mathrm{F}}} \leq \gamma\varepsilon,$$

where

$$\gamma := \frac{g(a^0)}{\|X\|_{\mathrm{F}}}$$

is the overall relative condition number of equation (1) at the particular solution X.

The estimate (9), (10) is valid only asymptotically. This means that the perturbations in the data must be small enough to ensure sufficient accuracy of the linear estimate. This disadvantage of the local estimate may be overcome using the techniques of non-local perturbation analysis, presented below.

3 Non-local perturbation analysis

The perturbed equation (2) may be written in the form

$$\mathcal{L}(\delta X) = \Phi_0(\delta A) + \Phi_1(\delta X, \delta A) + \Phi_2(\delta X, \delta A),$$

where

$$\begin{aligned}
\Phi_0(\delta A) &:= \delta A_1 + \sigma A_2^{\mathrm{H}} X^{-2}\delta A_2 + \sigma\delta A_2^{\mathrm{H}} X^{-2} A_2 + \sigma\delta A_2^{\mathrm{H}} X^{-2}\delta A_2,\\
\Phi_1(\delta X, \delta A) &:= -\sigma(A_2^{\mathrm{H}} X^{-2}\delta X X^{-1}\delta A_2 + A_2^{\mathrm{H}} X^{-1}\delta X X^{-2}\delta A_2\\
&+ \delta A_2^{\mathrm{H}} X^{-2}\delta X X^{-1} A_2 + \delta A_2^{\mathrm{H}} X^{-1}\delta X X^{-2} A_2\\
&+ \delta A_2^{\mathrm{H}} X^{-2}\delta X X^{-1}\delta A_2 + \delta A_2^{\mathrm{H}} X^{-1}\delta X X^{-2}\delta A_2),\\
\Phi_2(\delta X, \delta A) &:= \sigma(A_2 + \delta A_2)^{\mathrm{H}}(X^{-1}\delta X X^{-1})^2(A_2 + \delta A_2)
\end{aligned}$$

To obtain the above relations we use the approximation

$$(X + \delta X)^{-1} = X^{-1} - X^{-1}\delta X X^{-1}.$$

As a result we get the operator equation

$$\delta X = \Pi(\delta X, \delta A) := \Pi_0(\delta A) + \Pi_1(\delta X, \delta A) + \Pi_2(\delta X, \delta A), \tag{11}$$

where $\Pi_r = \mathcal{L}^{-1}(\Phi_r)$.

We shall show that under certain conditions on the F-norms δ_k of δA_k the operator $\Pi(\cdot, \delta A)$ maps a central ball $\mathcal{B}_\rho$ of diameter $\rho = f(\delta)$ into itself, where f is continuous and $f(0) = 0$. Hence according to the Schauder fixed point principle [15, 8] the operator equation (11) has a solution $\delta X \in \mathcal{B}_\rho$. Finally the estimate $\|\delta X\|_{\mathrm{F}} \leq f(\delta)$ is the desired non-local perturbation estimate for δ belonging to a certain set $\Omega \subset \mathbb{R}_+^2$ containing the origin.

Suppose that $\xi = \|\delta X\|_{\mathrm{F}} \leq \rho$, where ρ is a positive quantity. Then, after some calculations, we obtain the inequality

$$\|F(\delta X, \delta A)\|_{\mathrm{F}} \leq h(\rho, \delta) \quad := \quad a_0(\delta) + a_1(\delta)\rho + a_2(\delta)\rho^2$$

Here

$$\begin{aligned}
a_0(\delta) &:= g(\delta) + c_1\mu^2\delta_2^2, \\
a_1(\delta) &:= a_{11}\delta_2 + a_{12}\delta_2^2, \\
a_2(\delta) &:= a_{20} + a_{21}\delta_2 + a_{22}\delta_2^2, \\
\mu &:= \|X^{-1}\|_2.
\end{aligned}$$

and

$$\begin{aligned}
a_{11} &:= \mu \left\| L^{-1}\left(I_n \otimes \left(A_2^{\mathrm{H}} X^{-2}\right)\right)\right\|_2 + \mu^2 \left\| L^{-1}\left(I_n \otimes \left(A_2^{\mathrm{H}} X^{-1}\right)\right)\right\|_2 \\
&+ \mu \left\| L^{-1}\left(\left(X^{-1}A_2\right)^{\top} \otimes I_n\right) P_{n^2}\right\|_2 + \mu^2 \left\| L^{-1}\left(\left(X^{-2}A_2\right)^{\top} \otimes I_n\right) P_{n^2}\right\|_2, \\
a_{12} &:= 2c_1\mu^3, \\
a_{20} &:= \mu^4 \left\| L^{-1}\left(A_2^{\top} \otimes A_2^{\mathrm{H}}\right)\right\|_2, \\
a_{21} &:= \mu^4 \left\| L^{-1}\left(A_2^{\top} \otimes I_n\right) P_{n^2} + L^{-1}\left(I_n \otimes A_2^{\mathrm{H}}\right)\right\|_2, \\
a_{22} &:= c_1\mu^4.
\end{aligned}$$

The function h is a Lyapunov majorant for the operator Π, see [2, 9, 11]. The corresponding majorant equation $\rho = h(\rho, \delta)$ is equivalent to the quadratic equation

$$a_2(\delta)\rho^2 - (1 - a_1(\delta))\rho + a_0(\delta) = 0.$$

Consider the domain

$$\Omega := \left\{\delta \in \mathbb{R}_+^2 \ : \ a_1 + 2\sqrt{a_0 a_2} \leq 1\right\}. \tag{12}$$

If $\delta \in \Omega$ then the majorant equation $\rho = h(\rho, \delta)$ has a root

$$\rho(\delta) = f(\delta) := \frac{2a_0(\delta)}{1 - a_1 + \sqrt{(1 - a_1(\delta))^2 - 4a_0(\delta)a_2(\delta)}}. \tag{13}$$

Hence for $\delta \in \Omega$ the operator $\Pi(., \delta A)$ maps the set $\mathcal{B}_{f(\delta)}$ into itself, where

$$\mathcal{B}_r := \left\{x \in \mathbb{C}^{n^2} \ : \ \|x\|_2 \leq r\right\}$$

is the closed central ball of radius $r \geq 0$. Then according to Schauder fixed point principle there exists a solution $\delta X \in \mathcal{B}_{f(\delta)}$ of equation (11).

Thus we have the following result.

Theorem. *Let $\delta \in \Omega$, where Ω is given in (12). Then the non-local perturbation bound $\|\delta X\|_{\mathrm{F}} \leq f(\delta)$ is valid for equation (1), where $f(\delta)$ is determined by (13).*

4 Numerical example

Consider the complex matrix equation

$$X - A_1 - \sigma A_2^{\mathrm{H}} X^{-2} A_2 = 0$$

with matrices $X = I_3$, $A_1 = X - \sigma A_2^{\mathrm{H}} X^{-2} A_2$ and

$$A_2 = \begin{bmatrix} 0.22+0.05i & 0.68+0.01i & 0.52+0.42i \\ 0.05+0.53i & 0.93+0.38i & 0.83+0.69i \\ 0.68+0.67i & 0.38+0.07i & 0.03+0.59i \end{bmatrix}.$$

The perturbations in the data are taken as

$$\begin{aligned} \delta A_2 &= \delta X = 10^{(-k)} \begin{bmatrix} 1+i & 1+i & 1+i \\ 1+i & 1+i & 1+i \\ 1+i & 1+i & 1+i \end{bmatrix}, \\ \delta A_1 &= X + \delta X - \sigma(A_2 + \delta A_2)^{\mathrm{H}}(X + \delta X)^{-2}(A_2 + \delta A_2) - A_1. \end{aligned}$$

for $k = 10, 9, ..., 1$.

This problem was designed so as to have solutions $X = I_3$ and $X + \delta X = I_3 + \delta X$ of the unperturbed and perturbed equation respectively.

The perturbation $\|\delta X\|_F$ in the solution is estimated by the local bounds $\mathrm{est}_2(\delta)$, $\mathrm{est}_3(\delta)$ from Section 2 and the non-local bound (13), (12) from Section 3.

The case when the non-local estimate is not valid, since the existence condition $\delta \in \Omega$ is violated, is denoted by asterisk.

The results obtained for different values of k are shown at Table 1, for the equation with $\sigma = +1$ and at Table 2, when $\sigma = -1$. When k decreases from 10 to 1 the non-local estimate is slightly more pessimistic than the local bounds $\mathrm{est}_2(\delta)$, $\mathrm{est}_3(\delta)$. We also see that for this particular example the bound $\mathrm{est}_3(\delta)$ is superior to $\mathrm{est}_2(\delta)$.

Table 1. $\sigma = +1$

k	$\|\delta X\|_F$	est_2	est_3	$\rho(\delta)$ (13)
10	4.24×10^{-10}	1.92×10^{-8}	9.90×10^{-9}	9.90×10^{-9}
9	4.24×10^{-9}	1.92×10^{-7}	9.90×10^{-8}	9.90×10^{-8}
8	4.24×10^{-8}	1.92×10^{-6}	9.90×10^{-7}	9.90×10^{-7}
7	4.24×10^{-7}	1.92×10^{-5}	9.90×10^{-6}	9.90×10^{-6}
6	4.24×10^{-6}	1.92×10^{-4}	9.90×10^{-5}	9.91×10^{-5}
5	4.24×10^{-5}	1.92×10^{-3}	9.90×10^{-4}	9.93×10^{-4}
4	4.24×10^{-4}	1.92×10^{-2}	9.90×10^{-3}	1.01×10^{-2}
3	4.24×10^{-3}	1.92×10^{-1}	9.88×10^{-2}	1.28×10^{-1}
2	4.24×10^{-2}	1.87	9.70×10^{-1}	*
1	4.24×10^{-1}	$1.47\times10^{+1}$	8.15	*

Table 2. $\sigma = -1$

k	$\|\|\delta X\|\|_F$	est_2	est_3	$\rho(\delta)$ (13)
10	4.24×10^{-10}	7.93×10^{-9}	6.43×10^{-9}	6.43×10^{-9}
9	4.24×10^{-9}	7.93×10^{-8}	6.43×10^{-8}	6.43×10^{-8}
8	4.24×10^{-8}	7.93×10^{-7}	6.43×10^{-7}	6.43×10^{-7}
7	4.24×10^{-7}	7.93×10^{-6}	6.43×10^{-6}	6.43×10^{-6}
6	4.24×10^{-6}	7.93×10^{-5}	6.43×10^{-5}	6.43×10^{-5}
5	4.24×10^{-5}	7.63×10^{-4}	6.43×10^{-4}	6.43×10^{-4}
4	4.24×10^{-4}	7.93×10^{-3}	6.42×10^{-3}	6.49×10^{-3}
3	4.24×10^{-3}	7.90×10^{-2}	6.40×10^{-2}	7.16×10^{-2}
2	4.24×10^{-2}	7.60×10^{-1}	6.20×10^{-1}	*
1	4.24×10^{-1}	5.25	4.55	*

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ПЕРТУРБАЦИОНЕН АНАЛИЗ НА МАТРИЧНОТО УРАВНЕИЕ $\mathbf{X = A_1 + \sigma A_2^H X^{-2} A_2}$, $\sigma = \pm 1$

Вера Ангелова

Намерени са числа на обусловеност, локални и нелокални пертурбационни граници за комплексното матрично уравнение $X = A_1 + \sigma A_2^{\mathrm{H}} X^{-2} A_2$. Използвани са техники, основани на мажорантите на Ляпунов и принципите на фиксираната точка. Даден е илюстративен числен пример.