Perturbation analysis for the complex linear matrix equation $\alpha \mathbf{X} + \sigma \mathbf{A}^{\mathrm{H}}\mathbf{X}\mathbf{A} = \mathbf{I}$, $\alpha, \sigma = \pm 1$ *

Vera Angelova

Abstract

Condition numbers, local and non-local perturbation bounds are derived for the complex linear matrix equation $\alpha X + \sigma A^{\mathrm{H}} X A = I$, $\alpha, \sigma = \pm 1$. The technique used is based on Lyapunov majorants and fixed point principles. An illustrative numerical example is given.

Key words: perturbation analysis, matrix equations, discrete-time Lyapunov equation

MSC 2000: 15A24

1 Introduction and notation

In this paper a perturbation analysis for the complex linear matrix equation

$$F(X, A) := \alpha X + \sigma A^{\mathrm{H}} X A - I = 0, \quad \alpha, \sigma = \pm 1, \tag{1}$$

with $A, X \in \mathbb{C}^{n\times n}$ is presented. Here I is the identity $n \times n$ matrix and A^{H} represents the conjugate transpose of A, $A^{\mathrm{H}} = \bar{A}^{\top}$.

The existence of positive definite solution of (1) is considered in [1]. Necessary and sufficient conditions are derived and iterative methods for computing positive definite solution of the equation are investigated.

For $\sigma = 1$ equation (1) represents a particular case of the discrete-time Lyapunov matrix equation $A^{\mathrm{H}} X A - \alpha X = C$, $C = C^{\mathrm{H}}$, arising in the theory of discrete-time invariant linear control systems. Perturbation bounds for the real and complex continuous and discrete-time Lyapunov matrix equation, as well as for the descriptor Lyapunov equation are proposed in [2]. For analysis of particular classes of Lyapunov matrix equations see [3, 4, 5] and their references.

Throughout the paper the following notations are used: $\mathbb{R}^{n\times n}$ – the space of $n \times n$ real matrices; $\mathbb{C}^{n\times n}$ – the space of $n\times n$ complex matrices; $\mathbb{R}_+ = [0, \infty)$; $\mathrm{vec}(A) \in \mathbb{C}^{n^2}$ – the column-wise vector representation of the matrix $A \in \mathbb{C}^{n\times n}$; $\mathrm{Mat}(\mathbf{L}) \in \mathbb{C}^{n^2\times n^2}$ – the

*This research work was supported by the project No 010050 "Modelling of systems with parameter uncertainties and analysis of multiobjective optimization problems" of IIT - BAS.

matrix representation of the linear matrix operator $\mathbf{L} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$; $A \otimes B = [a_{pq}B]$ – the Kronecker product of the matrices $A = [a_{pq}]$ and B; $P_{n^2} \in \mathbb{C}^{n^2\times n^2}$ – the vec-permutation matrix such that $\mathrm{vec}(Y^\top) = P_{n^2}\mathrm{vec}(Y)$ for each $Y \in \mathbb{C}^{n\times n}$; $\|\cdot\|_2$ – the Euclidean or the spectral (or 2-) norm; $\|\cdot\|_{\mathrm{F}}$ – the Frobenius (or F-) norm in $\mathbb{C}^{n\times n}$; $\|\cdot\|$ – a replacement of either $\|\cdot\|_2$ or $\|\cdot\|_{\mathrm{F}}$; $z^{\mathcal{R}} \in \mathbb{R}^{2n}$ - the real version of $z \in \mathbb{C}^n$; $\Gamma^{\mathbb{R}} \in \mathbb{R}^{2n\times 2n}$ - the real version of $\Gamma \in \mathbb{C}^{n\times n}$; $\Theta(\Gamma, \Delta)$ - the matrix of the real version of the operator $z \to \Gamma z + \Delta\bar{z}$. The notation ':=' stands for 'equal by definition'.

The paper is organised as follows. In Section 2 a local perturbation analysis is presented. The perturbations in the data A and the solution X are estimated in terms of the Frobenius matrix norm $\|\cdot\|_{\mathrm{F}}$. Explicit expression for the condition number of X relative to perturbation in A is obtained. Rewriting (1) as an equivalent matrix equation for the perturbation in the solution, non-linear non-local bound is obtained in Section 3. The technique used is based on Lyapunov majorants and fixed point principles [6]. In Section 4 a numerical example demonstrates the effectiveness of the bounds proposed.

2 Local perturbation analysis

Let the coefficient matrix A be subject to perturbation δA, so that instead of the initial data the coefficient matrix is $A + \delta A$ and the solution of the perturbed equation

$$F(X + \delta X, A + \delta A) = 0 \tag{2}$$

is $X + \delta X$. Rewrite (2) as an equivalent equation for the perturbation δX in X

$$\begin{aligned} F(X + \delta X, A + \delta A) &= F(X, A) + F_X(X, A)(\delta X) + F_A(X, A)(\delta A) \\ &+ F_{\bar{A}}(X, A)(\delta\bar{A}) + G(X, A)(\delta X, \delta A), \end{aligned}$$

where $F_X(X, A)(Y) := \alpha Y + \sigma A^{\mathrm{H}} Y A : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is the partial Fréchet derivative of F in X calculated at the point (X, A). The term $F_A(X, A)(\delta A) + F_{\bar{A}}(X, A)(\delta\bar{A})$, where $F_A(X, A)(Y) = \sigma A^{\mathrm{H}} X Y$, $F_{\bar{A}}(X, A)(Y) = \sigma Y^{\mathrm{H}} X A$, is an additive but not homogeneous operator related to the partial Fréchet derivative of $F(X + \delta X, A + \delta A)$ in A. This is due to the fact that complex conjugation (and hence the map $A \to A^{\mathrm{H}}$) is not a linear operation. The term $G(\delta X, \delta A)$ contains second order terms in δA. The matrix of the operator $F_X(X, A)$ is $L = \alpha I_{n^2} + \sigma A^\top \otimes A^{\mathrm{H}}$. The eigenvalues of $F_X(X, A)$ are the eigenvalues of the matrix L and are equal to $\alpha + \sigma\lambda_i(A)\bar{\lambda}_j(A), \quad i, j = 1, 2, \ldots, n$. Here $\lambda_i(Z)$ is the eigenvalue of the matrix Z.

Hence the operator $F_X(X, A)$ and its matrix L are invertible iff

$$\sigma\lambda_i(A)\bar{\lambda}_j(A) \neq -\alpha. \tag{3}$$

Denote $x := \mathrm{vec}(\delta X), a := \mathrm{vec}(\delta A) \in \mathbb{C}^{n^2}$. Having in mind that $F(X, A) = 0$ and assuming that the operator $F_X(X, A)$ is invertible, i.e. the inequalities (3) hold true, we obtain

$$\delta X = -F_X^{-1} \circ F_A(\delta A) - F_X^{-1} \circ F_{\bar{A}}(\delta\bar{A}) + O(\|a\|^2), \; a \to 0$$

and

$$x = M_1 a + M_2 \bar{a} + O(\|a\|^2),\ a \to 0.$$

Here $M_1 = -L^{-1} L_A \in \mathbb{C}^{n^2 \times n^2}$ is the matrix of the operator $-F_X^{-1} \circ F_A$, $M_2 = -L^{-1} L_{\bar{A}} \in \mathbb{C}^{n^2 \times n^2}$ is the matrix of the operator $-F_X^{-1} \circ F_{\bar{A}}$ and $L_A = \sigma I_n \otimes (A^{\mathrm{H}} X)$, $L_{\bar{A}} = \sigma((XA)^\top \otimes I_n) P_{n^2} \in \mathbb{C}^{n^2 \times n^2}$ are the matrices of the operators F_A, $F_{\bar{A}}$ respectively.

For the real version of x is fulfilled [7]

$$x^{\mathcal{R}} = \Theta(M_1, M_2) a^{\mathcal{R}} + O(\|a^{\mathcal{R}}\|^2),\ a^{\mathcal{R}} \to 0,$$

where for $M_1 = M_{10} + iM_{11}$ and $M_2 = M_{20} + iM_{21}$

$$\Theta(M_1, M_2) := \begin{bmatrix} M_{10} + M_{20} & M_{21} - M_{11} \\ M_{11} + M_{21} & M_{10} - M_{20} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}. \tag{4}$$

For the matrices M_1, M_2 we obtain

$$M_1 = -\sigma L^{-1}(I_n \otimes (A^{\mathrm{H}} X)),\ M_2 = -\sigma L^{-1}(((XA)^\top \otimes I_n) P_{n^2}). \tag{5}$$

Denote $\delta := \|\delta A\|_{\mathrm{F}}$, $\xi := \|\delta X\|_{\mathrm{F}} = \|\mathrm{vec}(\delta X)\|_2 = \|x\|_2$. Since $\|x\|_2 = \|x^{\mathcal{R}}\|_2$ we find the local perturbation estimate $g(\delta)$

$$\xi \le g(\delta) := c\delta + O(\|\delta\|^2),\ \delta \to 0, \tag{6}$$

where $c = \|\Theta(M_1, M_2)\|_2$ is the absolute condition number and the matrix $\Theta(M_1, M_2)$ is given by (4). The relative condition number is computed from $\zeta := c\|A\|_{\mathrm{F}} / \|X\|_{\mathrm{F}}$.

Let $\delta = \varepsilon \|A\|_{\mathrm{F}}$, where $\varepsilon > 0$ (in floating point arithmetic the quantity ε may be taken as a multiple of the unit round off error of the machine). Then $g(\delta) = \varepsilon\, g(\|A\|_{\mathrm{F}})$. Hence the relative perturbation in the solution can be estimated as $\frac{\|\delta X\|_{\mathrm{F}}}{\|X\|_{\mathrm{F}}} \le \varepsilon \frac{g(\|A\|_{\mathrm{F}})}{\|X\|_{\mathrm{F}}}$.

The estimate (6) is valid only asymptotically. This means that the perturbations in the data must be small enough to ensure sufficient accuracy of the linear estimate. To overcome this disadvantage of the local estimate a non-local bound is obtained below, using the techniques of the non-local perturbation analysis.

3 Non-local perturbation analysis

The perturbed equation (2) may be written in the form $F_X(\delta X) = \Phi_0(\delta A) + \Phi_1(\delta X, \delta A)$, where

$$\begin{aligned} \Phi_0(\delta A) &:= -\sigma(A^{\mathrm{H}} X \delta A + \delta A^{\mathrm{H}} X A + \delta A^{\mathrm{H}} X \delta A), \\ \Phi_1(\delta X, \delta A) &:= -\sigma(A^{\mathrm{H}} \delta X \delta A + \delta A^{\mathrm{H}} \delta X A + \delta A^{\mathrm{H}} \delta X \delta A). \end{aligned}$$

As a result we get the operator equation

$$\delta X = \Pi(\delta X, \delta A) := \Pi_0(\delta A) + \Pi_1(\delta X, \delta A), \tag{7}$$

where $\Pi_r = F_X^{-1}(\Phi_r)$.

Suppose that $\xi \leq \rho$, where ρ is a positive quantity. Then, after some calculations, we obtain the inequality

$$\|F(\delta X, \delta A)\|_{\mathrm{F}} \leq h(\rho, \delta) := a_0(\delta) + a_1(\delta)\rho,$$

where

$$a_0(\delta) := g(\delta) + l\|X\|_2\delta^2, \quad a_1(\delta) := a_{11}\delta + l\delta^2,$$
$$a_{11} := \left\|L^{-1}\left(I_n \otimes A^{\mathrm{H}}\right)\right\|_2 + \left\|L^{-1}\left(\left(A^{\mathrm{H}} \otimes I_n\right) P_{n^2}\right)\right\|_2$$

and l is the Lyapunov norm [4, 6] of the inverse of the Lyapunov operator F_X

$$l = \|F_X^{-1}\|_{\mathrm{Lyap}} = \frac{1}{2}\|(L^{-1})^{\mathcal{R}}\mathrm{diag}(J_{n^2}, \ \ I_{n^2} - P_{n^2})\|_2,$$

where $J_{n^2} = I_{n^2} - P_{n^2}$. The use of the Lyapunov norm is in order to get tighter perturbation bounds.

The function h is a Lyapunov majorant for the operator Π, see [6]. The corresponding majorant equation $\rho = h(\rho, \delta)$ is linear and is equivalent to the equation $-(1 - a_1(\delta))\rho + a_0(\delta) = 0$.

Consider the domain

$$\Omega := \left\{\delta \in \mathbb{R}_+ \ : \ \delta < \frac{2}{a_{11} + \sqrt{a_{11}^2 + 4l}}\right\}. \tag{8}$$

If $\delta \in \Omega$ then the majorant equation $\rho = h(\rho, \delta)$ has a root

$$\rho(\delta) = f(\delta) := \frac{g(\delta) + l\|X\|_2\delta^2}{1 - a_{11}\delta - l\delta^2}. \tag{9}$$

Hence for $\delta \in \Omega$ the operator $\Pi(., \delta A)$ maps the set $\mathcal{B}_{f(\delta)}$ into itself, where $\mathcal{B}_r := \left\{x \in \mathbb{C}^{n^2} \ : \ \|x\|_2 \leq r\right\}$ is the closed central ball of radius $r \geq 0$. Then according to Schauder fixed point principle there exists a solution $\delta X \in \mathcal{B}_{f(\delta)}$ of equation (7).

Thus we have the following result.

Theorem. *Let $\delta \in \Omega$, where Ω is given in (8). Then the non-local perturbation bound $\|\delta X\|_{\mathrm{F}} \leq f(\delta)$ is valid for equation (1), where $f(\delta)$ is determined by (9).*

4 Numerical example

Consider the linear complex matrix equation $A^{\mathrm{H}}XA - X = I$. The data matrix A and the perturbations in the data are taken as

$$A = \begin{bmatrix} 0.22 + 0.05i & 0.68 + 0.01i & 0.52 + 0.42i \\ 0.05 + 0.53i & 0.93 + 0.38i & 0.83 + 0.69i \\ 0.68 + 0.67i & 0.38 + 0.07i & 0.03 + 0.59i \end{bmatrix}, \ \delta A = 10^{(-k)} \begin{bmatrix} 1+i & 1+i & 1+i \\ 1+i & 1+i & 1+i \\ 1+i & 1+i & 1+i \end{bmatrix}$$

for $k = 10, 9, ..., 1$. The perturbed and unperturbed problems are solved using the Matlab function *dlyap*.

The perturbation $\|\delta X\|_F$ in the solution is estimated by the local bound $g(\delta)$ (6) from Section 2 and the non-local bound (9), (8) from Section 3. The case when the non-local estimate is not valid since the existence condition $\delta \in \Omega$ is violated is denoted by asterisk. The results obtained for different values of k are shown at Table 1. When k decreases from 10 to 1 the non-local estimate is slightly more rough than the local bound $g(\delta)$.

References

[1] El- Sayed S.M., I.G. Ivanov, M.G. Petkov. In: *Applications of Mathematics in Engineering and Economics'26* Ed. B.I. Cheshankov and M.D. Todorov, Sofia, Heron Press, 2001, 142–146.

[2] Konstantinov M., P. Petkov, D.W. Gu, V. Mehrmann. In: Technical Report 98–15, Dept. of Engineering, Leicester Univ., Leicester, UK, 1998.

[3] Hewer G., C. Kenney. *SIAM J. Cont. Optim.*, **26**, 1988, 321–344.

[4] Konstantinov M., V. Mehrmann, P. Petkov. *Linear Algebra Appl.*, **312**, 2000, 35–71.

[5] Petkov P.Hr., D.W. Gu, M.M. Konstantinov, V. Mehrmann. In: Technical Report 2000-1, NICONET, Department of Electrical Eng., University of Leuven, 2000. http://www.win.tue.nl/wgs/niconet.html.

[6] Konstantinov M., V. Mehrmann, P. Petkov, D.W. Gu. In: Technical Report Prep. 760, Inst. Math., TU-Berlin, 2002. http://www.math.tu-berlin.de/preprints.

[7] Konstantinov M., P. Petkov, V. Mehrmann, D. Gu. *Proc. 30 Spring Conf. of Union of Bulgar. Mathematicians*, Borovetz, Bulgaria, 2001, 169–175.

Institute of Information Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., bl. 2
1113 Sofia, Bulgaria
e-mail: vangelova@iit.bas.bg

Table 1.

k	$\|\delta X\|_F$	$g(\delta)$ (6)	$\rho(\delta)$ (9)
10	3.01×10^{-10}	1.14×10^{-9}	1.14×10^{-9}
9	3.01×10^{-9}	1.14×10^{-8}	1.14×10^{-8}
8	3.01×10^{-8}	1.14×10^{-7}	1.14×10^{-7}
7	3.01×10^{-7}	1.14×10^{-6}	1.14×10^{-6}
6	3.01×10^{-6}	1.14×10^{-5}	1.14×10^{-5}
5	3.01×10^{-5}	1.14×10^{-4}	1.14×10^{-4}
4	3.00×10^{-4}	1.14×10^{-3}	1.15×10^{-3}
3	2.99×10^{-3}	1.14×10^{-2}	1.18×10^{-2}
2	2.91×10^{-2}	1.14×10^{-1}	1.55×10^{-1}
1	2.25×10^{-1}	1.14	*