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PERTURBATION BOUNDS FOR THE MATRIX EQUATION $C + \sum_{i=1}^{r} A_i X B_i + D X^s E = 0$

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Abstract

The paper proposes perturbation bounds for the solution of the matrix equation $C + \sum_{i=1}^{r} A_i X B_i + D X^s E = 0$, where r and $s \geq 2$ are positive integers. Local and nonlocal bounds are derived using the technique of Lyapunov majorants and fixed point principles.

Key words: perturbation analysis, matrix equations
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1. Introduction and notation. We consider the matrix equation

$$F(X, S) := C + \sum_{i=1}^{r} A_i X B_i + D X^s E = 0, \tag{1}$$

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where $X \in \mathbb{F}^{n\times n}$ is the $n \times n$ unknown matrix over the space of real $\mathbb{F} = \mathbb{R}$ or complex $\mathbb{F} = \mathbb{C}$ numbers and r, $s \geq 2$ are positive integers. The function $F(\cdot, S) : \mathbb{F}^{n\times n} \to \mathbb{F}^{p\times q}$ depends on the matrix collection $S := (C, D, E, A_1, B_1, A_2, B_2, \dots, A_r, B_r)$. Here $A_i \in \mathbb{F}^{p\times n}$, $B_i \in \mathbb{F}^{n\times q}$, $C \in \mathbb{F}^{p\times q}$, $D \in \mathbb{F}^{p\times n}$, $E \in \mathbb{F}^{n\times q}$, for $i = 1, 2, \dots, r$ are given data matrices and $pq = n^2$. The matrix $(2r + 3)$-tuple S depends on $n^2 + (r + 1)n(p + q)$ parameters – the elements of the matrices A_i, B_i, C, D, E.

The nonlinear matrix equation $X + A^*\mathcal{F}(X)A = Q$ is considered in [1]. Sufficient conditions for the existence and uniqueness of a positive semidefinite solution are derived and upper perturbation bounds are proposed. An iteration method for solving this equation is considered in [2] and an estimate for the range of convergence of a certain iteration scheme is given. In [3] the solution of the quadratic matrix equation $AX^2+BX+C = 0$ by Newton method with exact line searches is considered. A condition number is derived and the computation of the backward error of an approximate solution is estimated.

In this paper a complete perturbation analysis of (1) is made. It allows to derive condition and accuracy estimates for the computed solution when a numerically stable algorithm is applied to compute the solution.

Throughout the paper the following notations are used: $\mathbb{N}$ is the set of natural numbers; I_n is the identity $n \times n$ matrix; $\mathrm{vec}(A) \in \mathbb{F}^{n^2}$ is the column-wise vector representation of the matrix $A \in \mathbb{F}^{n\times n}$, where $\mathbb{F}^n = \mathbb{F}^{n\times 1}$; $\mathrm{Mat}(\mathcal{L}) \in \mathbb{F}^{n^2\times n^2}$ is the matrix representation of the linear matrix operator $\mathcal{L} : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$; $A \otimes B = [A(k, l)B]$ is the Kronecker product of the matrices $A = [A(k, l)]$ and B; $\|\cdot\|$ is the vector or a matrix norm; $\|\cdot\|_2$ is the Euclidean vector or the spectral matrix norm; $\|\cdot\|_{\mathrm{F}}$ is the Frobenius matrix norm.

The notation ':=' stands for 'equal by definition.'

2. Statement of the problem. Consider for simplicity the case $\mathbb{F} = \mathbb{R}$. The case $\mathbb{F} = \mathbb{C}$ is treated similarly.

Denote by $F_Z(X, S) : \mathbb{F}^{k\times\nu} \to \mathbb{F}^{p\times q}$ the partial Fréchet derivative of F in the corresponding $k \times l$ matrix argument $Z \in \mathcal{S} := \{C, D, E, A_1, B_1, A_2, B_2, \dots, A_r, B_r\}$, computed at the point (X, S).

We assume that equation (1) has a solution X such that the linear operator $F_X := F_X(X, S) : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$ is invertible, where $F_X(X, S)$ is the Fréchet derivative of F in X at the point (X, S). Then according to the implicit function theorem [4,5] the solution X is isolated, i.e. there exists $\varepsilon > 0$ such that equation (1) has no other solutions $\tilde{X}$ with $\|\tilde{X} - X\| < \varepsilon$.

The perturbation problem for equation (1) is stated as follows. Let the matrices from $\mathcal{S}$ be perturbed as $A_i \to A_i + \delta A_i$, $B_i \to B_i + \delta B_i$, $C \to C + \delta C$, $D \to D + \delta D$, $E \to E + \delta E$. If some of the above matrices are not perturbed then the corresponding perturbations are assumed to be zero.

Denote by $S + \delta S$ the perturbed collection S, in which each matrix $Z \in \mathcal{S}$ is replaced by $Z + \delta Z$. The perturbed equation is

$$(2) \quad F(X + \delta X, S + \delta S) = C + \delta C + \sum_{i=1}^{r}(A_i + \delta A_i)(X + \delta X)(B_i + \delta B_i) + (D + \delta D)(X + \delta X)^s(E + \delta E) = 0,$$

where $X + \delta X$ is the solution of (2) and δX is the perturbation in the solution X of equation (1).

In general some of the coefficient matrices from $\mathcal{S}$ may not be perturbed. When for e.g., some of the matrices are unit matrices then the corresponding perturbations are assumed to be zero. Denote by $\mathcal{S}^* := \{Z_1, Z_2, \ldots, Z_l\}$, $l \leq 3+2r$, the set of all matrices from $\mathcal{S}$, which are perturbed.

Since the operator F_X is assumed to be invertible, equation (2) has a unique solution $X+\delta X$ in the neighbourhood of X for perturbations δS sufficiently small. The elements of the perturbation δX are analytical functions of the elements of the perturbation δS in the data S.

Denote by $\delta := \left[\, \delta_{Z_1} \;\; \delta_{Z_2} \;\; \ldots \;\; \delta_{Z_l} \,\right]^\top \in \mathbb{R}^l_+$ the vector of nonzero absolute Frobenius norms $\delta_Z := \|\delta Z\|_{\mathrm{F}}$ of the perturbations δZ in the data matrices $Z \in \mathcal{S}^*$.

The perturbation analysis problem for the matrix equation (1) is to find a bound

$$(3) \qquad \delta_X \leq f(\delta),\ \delta \in \Omega \subset \mathbb{R}^l_+$$

for the perturbation $\delta_X := \|\delta X\|_{\mathrm{F}}$. Here Ω is a certain set and f is a continuous function, nondecreasing in each of its arguments and satisfying $f(0) = 0$. The inclusion $\delta \in \Omega$ guarantees that the perturbed equation (2) has a unique solution $X+\delta X$ in a neighbourhood of the unperturbed solution X such that the elements of δX are analytic functions of the elements of the matrices δZ, $Z \in \mathcal{S}^*$.

3. Local perturbation analysis. The perturbed equation (2) may be written as an equivalent equation for the perturbation δX in X. One has $F(X + \delta X, S+\delta S) = F(X,S) + F_X(X,S)(\delta X) + \sum_{Z\in\mathcal{S}^*} F_Z(\delta Z) + G(X,S)(\delta X, \delta S) = 0$, where $F_Z(\cdot) = F_Z(X,S)(\cdot)$ is the partial Fréchet derivative of the function $F(\cdot,\cdot)$ in the argument $Z \in \mathcal{S}^*$ evaluated at the point (X,S). Explicit expressions for the Fréchet derivatives of the matrix valued functions $X \to X^s$ are given in [6,7]. The matrix $G(\delta X, \delta S)$ contains second and higher order terms in δX, δS,

$$\begin{aligned} G(\delta X, \delta S) = & \sum_{i=1}^{r}(A_i\delta X\delta B_i + \delta A_i\delta X B_i + \delta A_i X\delta B_i + \delta A_i\delta X\delta B_i) \\ & + DN(X,\delta X)E + D(M(X,\delta X) + N(X,\delta X))\delta E \\ & + \delta D(M(X,\delta X) + N(X,\delta X))E + \delta D(X+\delta X)^s\delta E + \mathrm{O}(u^3),\, u \to 0, \end{aligned}$$

where

$$
\begin{aligned}
(4)\qquad M(X,\delta X) &:= \sum_{i=1}^{s} X^{i-1}\delta X X^{s-i},\\
N(X,\delta X) &:= \sum_{i=2}^{s} X^{i-2}\delta X^2 X^{s-i} + \sum_{j=3}^{s}\sum_{i=j}^{s} X^{i-j}\delta X X^{j-2}\delta X X^{s-i},\\
u &:= \delta_X + \delta.
\end{aligned}
$$

In what follows it is supposed that the asymptotic estimates of the form $\mathrm{O}(u^k)$, $k = 1, 2$, are valid for $u \to 0$.

Define the linear operators

$$
\begin{aligned}
\mathcal{L}(.) &:= F_X(X,S)(.) : \mathbb{F}^{n\times n} \to \mathbb{F}^{p\times q},\\
\mathcal{L}_Z(.) &:= F_Z(X,S)(.) : \mathbb{F}^{k\times \nu} \to \mathbb{F}^{p\times q} \text{ by}
\end{aligned}
$$

$$
\begin{aligned}
(5)\qquad \mathcal{L}(H) &= \sum_{i=1}^{r} A_i H B_i + DM(X,H)E,\ \mathcal{L}_C(H) = H,\\
\mathcal{L}_{A_i}(H) &= HXB_i,\ L_{B_i}(H) = A_i X H,\\
\mathcal{L}_D(H) &= HX^sE,\ L_E(H) = DX^sH.
\end{aligned}
$$

Then

$$
F(X+\delta X, S+\delta S) = F(X,A) + \mathcal{L}(\delta X) + \sum_{Z\in\mathcal{S}^*} \mathcal{L}_Z(\delta Z) + G(\delta X,\delta S) = 0.
$$

The matrix representations L, L_Z of the operators $\mathcal{L}$, $\mathcal{L}_Z$ are

$$
\begin{aligned}
(6)\qquad L &:= \sum_{i=1}^{r} B_i^\top \otimes A_i - \sum_{i=1}^{s}((X^{s-i}E)^\top \otimes DX^{i-1}),\ L_{A_i} := (XB_i)^\top \otimes I_{pn},\\
L_{B_i} &:= I_{nq}\otimes A_iX, \mathcal{L}_C := I_{pq},\ L_D := (X^sE)^\top \otimes I_{pn},\ L_E := I_{nq}\otimes DX^s.
\end{aligned}
$$

Since the operator $\mathcal{L}$ is invertible, i.e. if its matrix representation L is nonsingular, then the perturbed equation (2) may be rewritten as an equivalent matrix equation [8,9]

$$
\begin{aligned}
(7)\qquad \delta X &= \Pi(\delta X,\delta S)\\
&:= -\sum_{Z\in\mathcal{S}^*} \mathcal{L}^{-1}\circ \mathcal{L}_Z(\delta S) - \mathcal{L}^{-1}\circ G(\delta X,\delta S) + \mathrm{O}(u^2), u \to 0.
\end{aligned}
$$

Rewrite the matrix equation (7) in a vector form applying the vec operation to the first-order terms $\mathrm{O}(u)$ and having in mind that $F(X,S)=0$:

$$\text{(8)} \qquad \mathrm{vec}(\delta X) = \sum_{Z\in\mathcal{S}^*} W_Z \mathrm{vec}(\delta Z) + \mathrm{O}(u^2),\ W_Z := -L^{-1}L_Z,\ Z\in\mathcal{S}^*.$$

Equation (8) makes it possible to obtain bounds in terms of absolute or relative condition numbers

$$\delta_X = \|\delta X\|_{\mathrm{F}} = \|\mathrm{vec}(\delta X)\|_2 \quad\leq\quad \mathrm{est}_1(\delta) + \mathrm{O}(u^2) := \sum_{Z\in\mathcal{S}^*} K_Z\delta_Z + \mathrm{O}(u^2), u\to 0.$$

The quantities

$$\text{(9)} \qquad K_Z = \|W_Z\|_2,\ Z\in\mathcal{S}^*$$

are the absolute condition numbers of (1) relative to perturbations in the matrix coefficients $Z\in\mathcal{S}^*$. The corresponding relative condition numbers are $k_Z := K_Z\|Z\|/\|X\|$.

Relation (8) also gives $\delta_X \leq \mathrm{est}_2(\delta) + \mathrm{O}(u^2) := \|W\|_2\|\delta\|_2 + \mathrm{O}(u^2), u\to 0$, where $W := \left[\ W_{Z_1}, W_{Z_2}, \dots, W_{Z_l}\ \right]$. The bounds $\mathrm{est}_1(\delta)$ and $\mathrm{est}_2(\delta)$ are alternative; which one is better depends on the particular value of δ.

There is a third bound, which is always less than or equal to est_1, namely $\delta_x \leq \mathrm{est}_3(\delta) + \mathrm{O}(u^2) := \sqrt{\delta^\top R\delta} + \mathrm{O}(u^2), u\to 0$, where R is the $l\times l$ matrix with elements $\|W_{Z_i}^\top W_{Z_j}\|_2$, $i,j=1,2,\dots,l$. Since $\mathrm{est}_3(\delta)\leq\mathrm{est}_1(\delta)$, we have the following result.

Theorem 1. The following first-order overall bound is valid for the norm of the vectorization $\mathrm{vec}(\delta X)$ of the perturbation δX in the solution X

$$\text{(10)} \qquad \delta_X \leq \mathrm{est}(\delta) + \mathrm{O}(u^2), u\to 0,\ \mathrm{est}(\delta) := \min\left\{\mathrm{est}_2(\delta), \mathrm{est}_3(\delta)\right\}.$$

The local bound (10) is valid only asymptotically, for $\delta\to 0$. This means that the perturbation in the data must be small enough to ensure sufficient accuracy of the local bound. Unfortunately, no one can say how small must be the perturbation for a given problem. For some critical values of the perturbations in the data matrices the equation may not have a solution, but the local estimate will still produce a 'bound'. The disadvantages of the local bound are overcome in the nonlocal perturbation bounds which have the following properties: a) The nonlocal bound is valid for data perturbations included in a given a priori prescribed domain. This guarantees that the perturbed equation has a unique solution in a neighbourhood of the unperturbed solution. b) The nonlocal bound is rigorous. A disadvantage of the nonlocal bound is that it may not exist or may be rather pessimistic in some cases.

4. Nonlocal perturbation analysis. We shall consider the expansion of $(A+E)^s$ for small matrices E keeping only the terms of first and second order in E.

For $s=2$ we have $(A+E)^2 = A^2 + EA + AE + E^2$.

For $s=3$ it is fulfilled $(A+E)^3 = A^3 + EA^2 + AEA + A^2E + E^2A + AE^2 + EAE + \mathrm{O}(\|E\|^3)$.

When $s>3$ we have $(A+E)^s = A^s + \Gamma(A,E)$, where $\Gamma(A,E) = M(A,E) + N(A,E) + \mathrm{O}(\|E\|^3)$, with M and N given in (4).

The perturbed equation (2) may be written in the form $\mathcal{L}(\delta X) = \Phi_0(\delta S) + \Phi_1(\delta X, \delta S) + \Phi_2(\delta X, \delta S) + \mathrm{O}(\|\delta X\|^3)$, where

$$\begin{aligned}
\Phi_0(\delta S) \quad &:= \quad -\delta C - \sum_{i=1}^{r}(A_i X \delta B_i + \delta A_i X B_i + \delta A_i X \delta B_i) \\
&\quad - \delta D X^s E - D X^s \delta E - \delta D X^s \delta E, \\
\Phi_1(\delta X, \delta S) \quad &:= \quad -\sum_{i=1}^{r}(A_i \delta X \delta B_i + \delta A_i \delta X B_i + \delta A_i \delta X \delta B_i) \\
&\quad - DM(X,\delta X)\delta E - \delta D M(X,\delta X) E - \delta D M(X,\delta X)\delta E, \\
\Phi_2(\delta X, \delta S) \quad &:= \quad -(D+\delta D) N(X,\delta X)(E+\delta E).
\end{aligned}$$

As a result, neglecting the terms of third and higher order in δX, the operator equation

$$\delta X \quad = \quad \Theta(\delta X, \delta S) := \Theta_0(\delta S) + \Theta_1(\delta X, \delta S) + \Theta_2(\delta X, \delta S) \tag{11}$$

is obtained, where $\Theta_i = \mathcal{L}^{-1}(\Phi_i),\ i = 0,1,2$.

Suppose that $\delta_X \le \rho$, where ρ is a positive quantity. After standard computations the following inequality is obtained:

$$\delta_X \le a_2(\delta)\rho^2 + a_1(\delta)\rho + a_0(\delta), \tag{12}$$

where

$$a_0(\delta) \quad := \quad \mathrm{est}(\delta) + \|L^{-1}\|_2 \left(\sum_{i=1}^{r} \|X\|_2 \delta_{A_i} \delta_{B_i} + \|X^s\|_2 \delta_D \delta_E \right),$$

$$
\begin{aligned}
a_1(\delta) :=& \sum_{i=1}^{r}\Big(\|L^{-1}(I\otimes A_i)\|_2\delta_{B_i} + \|L^{-1}\|_2\|B_i^{\top}\otimes I\|_2\delta_{A_i} + \|L^{-1}\|_2\delta_{A_i}\delta_{B_i}\Big)\\
&+ \|L^{-1}\|_2\left\|\sum_{i=1}^{s}(X^{s-i})^{\top}\otimes X^{i-1}\right\|_2\delta_D\delta_E\\
&+ \|L^{-1}\|_2\left\|\sum_{i=1}^{s}(X^{s-i}E)^{\top}\otimes X^{i-1}\right\|_2\delta_D\\
&+ \left\|\sum_{i=1}^{s}L^{-1}(X^{s-i})^{\top}\otimes DX^{i-1}\right\|_2\delta_E,
\end{aligned}
$$

$$
\begin{aligned}
a_2(\delta) :=& \sum_{j=3}^{s}\left\|\sum_{i=j}^{s}L^{-1}(X^{s-i}E)^{\top}\otimes DX^{i-j}\right\|_2\|X^{j-2}\|_2\\
&+ \left\|\sum_{i=2}^{s}L^{-1}(X^{s-i}E)^{\top}\otimes DX^{i-2}\right\|_2\\
&+ \sum_{i=2}^{s}\left\|L^{-1}(I\otimes DX^{i-2})\right\|_2\|X^{s-i}\|_2\delta_E\\
&+ \sum_{j=3}^{s}\sum_{i=j}^{s}\left\|L^{-1}(I\otimes DX^{i-j})\right\|_2\|X^{s-i}\|_2\|X^{j-2}\|_2\delta_E\\
&+\|L^{-1}\|_2\sum_{j=3}^{s}\sum_{i=j}^{s}\left\|(X^{s-i}E)^{\top}\otimes I\right\|_2\|X^{j-2}\|_2\|X^{i-j}\|_2\delta_D\\
&+ \|L^{-1}\|_2\sum_{i=2}^{s}\Big(\left\|(X^{s-i}E)^{\top}\otimes I\right\|_2 + \|X^{s-i}\|_2\delta_E\Big)\|X^{i-2}\|_2\delta_D\\
&+\|L^{-1}\|_2\sum_{j=3}^{s}\sum_{i=j}^{s}\left\|X^{j-i}\|_2\|X^{i-j}\right\|_2\|X^{j-2}\|_2\delta_D\delta_E.
\end{aligned}
\tag{13}
$$

The Lyapunov majorant for the operator Θ (see [8, 10]) such that

$$\|\Theta(\delta X,\delta S)\|_{\mathrm{F}} \le h(\rho,\delta),$$

is

$$h(\rho,\delta) = a_2(\delta)\rho^2 + a_1(\delta)\rho + a_0(\delta).$$

Thus the majorant equation $h(\rho,\delta) = \rho$ for determining the nonlocal bound $\rho = \rho(\delta)$ for δ_X is quadratic:

$$a_2(\delta)\rho^2 - (1 - a_1(\delta))\rho + a_0(\delta) = 0. \tag{14}$$

Suppose that $\delta \in \Omega$, where

$$\Omega := \left\{\delta \geq 0 : a_1(\delta) + 2\sqrt{a_0(\delta)a_2(\delta)} \leq 1\right\} \subset \mathbb{R}^l_+. \tag{15}$$

The inclusion $\delta \in \Omega$ guarantees that equation (14) has a root

$$\rho = f(\delta) := \frac{2a_0(\delta)}{1 - a_1(\delta) + \sqrt{(1 - a_1(\delta))^2 - 4a_0(\delta)a_2(\delta)}}. \tag{16}$$

Hence, the operator $\Theta(\cdot, \delta S)$ maps the closed convex ball $\mathcal{B}(\delta) := \{H \in \mathbb{F}^{n\times n} : \|H\|_{\mathrm{F}} \leq f(\delta), \delta \in \Omega\} \subset \mathbb{F}^{n\times n}$ into itself. According to the Schauder fixed point principle there exists a solution $\delta X \in \mathcal{B}(\delta)$ of equation (14), for which

$$\delta_X = \|\delta X\|_{\mathrm{F}} \leq f(\delta) + \mathrm{O}(\delta^3),\ \ \delta \in \Omega. \tag{17}$$

Thus the following result holds true.

Theorem 2. Let $\delta \in \Omega$, where Ω is given in (15). Then the nonlocal perturbation bound $\delta_X \leq f(\delta)$ is valid for equation (1), where $f(\delta)$ is determined by relations (16) and (13).

The nonlocal bound requires the calculation of the value $\|X^s\|$. In many cases it is more convenient to use the bounds for this quantity as follows [11], Appendix A in [12]:

- Based on the Jordan decomposition of X

$$\|X^s\| \leq \begin{cases} \mathrm{cond}(V)(1+\alpha)^s, & 0 \leq s \leq n-1 \\ \mathrm{cond}(V)\sum_{i=0}^{n-1}\binom{i}{s}\alpha^{s-i}, & s \geq n, \end{cases}$$

  where $X = VJV^{-1} \in \mathbb{R}^{n\times n}$ is the Jordan decomposition of X and $\alpha = \|J\|$.

- Based on the Schur decomposition of X

$$\|X^s\| \leq \begin{cases} (\lambda+\nu)^s, & 0 \leq s \leq n-1 \\ \sum_{i=0}^{n-1}\binom{i}{s}\nu^i\lambda^{s-i}, & s \geq n, \end{cases}$$

where $X = QTQ^{\mathrm{H}} \in \mathbb{R}^{n\times n}$ is the Schur decomposition of X, the matrix Q is unitary, T is upper triangular matrix, $\Lambda = \mathrm{diag}(T)$, $\lambda = \|\Lambda\|$ and $\nu = \|T - \Lambda\|$.

5. Concluding remarks. In this paper a perturbation analysis of the matrix equation $C + \sum_{i=1}^{r} A_i X B_i + DX^s E = 0$, $s, r \in \mathbb{N}$ is presented. Condition numbers, local and nonlocal perturbation bounds are derived. The local bound gives satisfactory results for small perturbations in the data. The nonlocal bound is slightly more pessimistic but holds when the perturbation in the data belongs to a preliminary defined domain of applicability of the bound. Numerical examples demonstrate the effectiveness of the proposed bounds.

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