New Upper Bounds for the CALE: A Singular Value Decomposition Approach

Svetoslav G. Savov and Ivan P. Popchev

Abstract: Motivated by the fact that upper solution bounds for the continuous Lyapunov equation are valid under some very restrictive conditions, an attempt is made to extend the set of Hurwitz matrices for which such bounds are applicable. It is shown that the matrix set for which solution bounds are available is only a subset of another stable matrices set. This helps to loosen the validity restriction. The new bounds are illustrated by examples.

Keywords: Continuous Lyapunov equation, Hurwitz matrix, singular value decomposition, solution upper bounds.

1. INTRODUCTION

The continuous algebraic Lyapunov equation (CALE) has been widely used in engineering theory. Although there are many numerical algorithms used to obtain a solution, sometimes only its estimate is needed. For instance, one can estimate a stability margin for real polynomials using some of the available bounds [6]. An estimate based approach to study robust stability and performance analysis of uncertain stochastic systems was suggested in [1]. Lyapunov equations have been used in atmospheric science applications [8], where a distinguishing property of the coefficient matrix A is its large dimension, which leads to impossibility for direct solution. Therefore, for practical purposes it is desirable to determine reasonable estimates.

The estimation problem for the CALE has attracted considerable attention over the past three decades. Excellent summaries on this topic were given in [4,5], comprising various lower and upper bounds for the largest eigenvalue, the trace, the determinant, the solution itself, etc. In practical applications, especially for stability analysis upper bounds are desired.

In most cases, the existing upper bounds are valid under some, unfortunately, very restrictive assumptions, e.g., $A^T + A$ must be negative definite. Since the stability of A does not guarantee this requirement, the respective estimates are useless for a large set of stable matrices.

This fact motivated the authors of this paper to investigate the conditions under which it is possible to have valid upper bounds for the solution of the CALE in cases when the existing bounds do not hold.

In other words, the results presented here extend the set of stable matrices A, for which upper bounds for the largest eigenvalue, the trace, and the solution matrix, are valid. This paper is organized as follows. The best known solution upper scalar bounds, when $A^T + A$ is a negative definite matrix and a recent result (based on the usage of similarity transformation) guaranteeing upper bounds validity for any Hurwitz matrix A are recalled in Section 2. Section 3 contains the main results consisting in extending the set of Hurwitz matrices for which upper matrix, maximum eigenvalue and trace bounds for the solution are valid (Theorem1 and Lemma 2). This is achieved via the singular value decomposition of matrix A. Some computational aspects of the suggested approach are discussed as well. The applicability of the proposed bounds is illustrated by several examples in Section 4, where the new bounds are compared with the best known upper solution estimates.

2. NOTATIONS AND PRELIMINARIES

In what follows, the given below notations will be used. H is the set of Hurwitz (negative stable) matrices, the symmetric part of matrix A is $A_s = 0.5(A^T + A)$ and $A > B, A \geq B$ means that $A - B$ is a positive definite and positive semi-definite matrix, respectively. The eigenvalues (when real) of a $n \times n$ matrix A are denoted $\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_n(A)$, $\lambda_1(A_s) \equiv \mu(A)$ is the

Manuscript received July 18, 2006; revised April 23, 2007, October 1, 2007, and November 19, 2007; accepted December 26, 2007. Recommended by Editorial Board member Yoshito Ohta under the direction of Editor Jae Weon Choi. This work is supported by the Bulgarian Academy of Sciences under grant No. 010077/2007.

Svetoslav G. Savov and Ivan P. Popchev are with the Institute of Information Technologies, Bulgarian Academy of Sciences, Acad. G. Bonchev str., bl. 2, 1113 Sofia, Bulgaria (e-mails: savovsg@ yahoo.com, ipopchev@iit.bas.bg).

matrix measure of A. The maximum and minimum singular values of matrix A are $\sigma_1(A)$ and $\sigma_n(A)$, respectively. The maximum real part of the eigenvalues of matrix A is $\kappa(A)$ and I is the identity matrix.

Consider the CALE

$$A^T P + PA + Q = 0, A \in \mathrm{H}, Q > 0, \quad (1)$$

with respect to the unknown (positive definite) matrix P. If $\mu(A) < 0$, the best known upper scalar solution bounds are:

$$\lambda_1(P) \le l_0 = \frac{1}{2}\lambda_1(-QA_s^{-1}) \ [9], \quad (2)$$

$$\mathrm{tr}(P) \le t_0 = -\frac{1}{2}\sum_1^n [\lambda_i(Q)/\lambda_i(A_s)] \ [3]. \quad (3)$$

Let T be a nonsingular matrix, denote $\tilde{A} = TAT^{-1}$ and define the set $\mathrm{H}^- \equiv \{\tilde{A} : \mu[\tilde{A}] < 0\}$. Equation (1) can be written in a modified form as:

$$\begin{aligned} &\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{Q} = 0, \\ &\tilde{P} = T^{-T} P T^{-1}, \\ &\tilde{Q} = T^{-T} Q T^{-1}. \end{aligned} \quad (4)$$

For any given matrix $A \in \mathrm{H}, A \notin \mathrm{H}^-$, there exists matrix T, such that $\tilde{A} \in \mathrm{H}^-$. This well known fact is used in [1] to apply bounds (2) and (3) for $\tilde{P}$ in (4) and then to get estimates for P in (1).

It is also possible to obtain a matrix bound. Consider the modified equation in (4). Let for some nonsingular matrix T, $\tilde{A} \in \mathrm{H}^-$, i.e., $\tilde{A}_s < 0$. Consider the positive scalar η defined as follows:

$$\begin{aligned} \eta &= \frac{1}{2}\lambda_1\{-Q[(T^T TA)_s]^{-1}\} \\ &= \lambda_1[-Q(A^T T^T T + T^T TA)^{-1}] \\ &= \lambda_1\{-Q[T^T(T^{-T} A^T T^T + TAT^{-1})T]^{-1}\} \\ &= \frac{1}{2}\lambda_1[\tilde{Q}(-\tilde{A}_s)^{-1}] = \frac{1}{2}\lambda_1[(-\tilde{A}_s)^{-1/2}\tilde{Q}(-\tilde{A}_s)^{-1/2}]. \end{aligned}$$

Since $2\eta I \ge (-\tilde{A}_s)^{-1/2}\tilde{Q}(-\tilde{A}_s)^{-1/2}$ and taking into account (4), one gets the following matrix inequality

$$\begin{aligned} 0 \ge 2\eta\tilde{A}_s + \tilde{Q} &= \eta(\tilde{A}^T + \tilde{A}) + \tilde{Q} - \tilde{A}^T\tilde{P} - \tilde{P}\tilde{A} - \tilde{Q} \\ &= \tilde{A}^T(\eta I - \tilde{P}) + (\eta I - \tilde{P})\tilde{A}. \end{aligned}$$

Matrix $\tilde{A}$ is Hurwitz and in accordance with Lyapunov's stability theory $\eta I - \tilde{P}$ must be a positive semi-definite matrix, which yields the upper matrix bound

$$P \le \eta T^T T, \eta = \frac{1}{2}\lambda_1\{-Q[(T^T TA)_s]^{-1}\}, \quad (5)$$

and then scalar bounds for $\lambda_1(P)$ and tr(P) are easily obtained.

The estimation problem for P has three important aspects: (i) restrictions on matrix A, (ii) computational burden, and (iii) tightness of the bounds. Bounds, based on Equation (4) eliminate problem (i), but require the determination of matrix T. The selection of how to obtain the tightest bound is an open and difficult question.

Matrix T is obtained by some additional computational procedure and in this sense $\bar{P} = T^T T$, $\mu(\bar{P}A) < 0$, is said to be an external Lyapunov matrix (ELM) for A. An internal Lyapunov matrix (ILM) is a matrix which can be defined entirely in terms of A. E.g., if $A \in \mathrm{H}^-$, then $\bar{P} = A^T A$ is an ILM for A.

This paper is an attempt to overcome to a certain extent the above mentioned difficulties concerning bounds based on ELM. This is closely related with the definition of an extension $\tilde{\mathrm{H}}$ of the conservative set H^- in the sense that, if $A \notin \mathrm{H}^-$, but $A \in \tilde{\mathrm{H}}$, there exists ILM for A. This will help to avoid the computation of an ELM which, as the order of A increases, may cause difficulties comparable with the solution of the CALE itself and thus make the respective bounds practically inapplicable.

3. MAIN RESULTS

Using the singular value decomposition (s.v.d.) of the coefficient matrix in (1), i.e., $A = U\Sigma V^T$, $UU^T = VV^T = I$ and Σ is a positive diagonal matrix containing the singular values of A, it is always possible to present it as a product of two matrices as follows

$$A = FP_1 = P_2F, F = UV^T, P_1^2 = A^T A, P_2^2 = AA^T.$$

The s.v.d. of the transformed matrix $\tilde{A}$ results in the following respective representations:

$$\begin{aligned} &\tilde{A} = TAT^{-1} = \tilde{U}\tilde{\Sigma}\tilde{V}^T = \tilde{F}\tilde{P}_1 = \tilde{P}_2\tilde{F}, \\ &\tilde{F} = \tilde{U}\tilde{V}^T, \tilde{P}_1^2 = \tilde{A}^T\tilde{A}, \tilde{P}_2^2 = \tilde{A}\tilde{A}^T, \end{aligned} \quad (6)$$

with $\tilde{U}\tilde{U}^T = \tilde{V}\tilde{V}^T = I$ and $\tilde{\Sigma}$ is a positive diagonal matrix. Define the matrix set

$$\tilde{\mathrm{H}} \equiv \{\tilde{A} : \tilde{F} \in \mathrm{H}\}.$$

Theorem 1: Denote $S = T^T \tilde{P}_1 T$. For any matrix $A \in \mathrm{H}$ and nonsingular T, one has

(a) $\lambda_1[(T^T TA)_s S^{-1}] \geq \kappa[\tilde{F}]$,

(b) $\mathrm{H}^- \subseteq \tilde{\mathrm{H}}$,

(c) $\tilde{A} \in \tilde{\mathrm{H}} \Leftrightarrow \mu(SA) < 0$.

Proof: Having in mind (6), one gets

$$\tilde{P}_1^{-1/2} \tilde{A} \tilde{P}_1^{-1/2} = \tilde{P}_1^{-1/2} \tilde{F} \tilde{P}_1^{1/2} = X.$$

Assertion (a) is proved applying the well known inequality $\mu(Y) \geq \kappa(Y)$ valid for any matrix Y [2] to X, which results in

$$\begin{aligned}\mu(X) &= \lambda_1[\tilde{P}_1^{-1/2} T^{-T} (T^T TA)_s T^{-1} \tilde{P}_1^{-1/2}] \\ &= \lambda_1[(T^T TA)_s S^{-1}] = \mu[\tilde{P}_1^{-1/2} \tilde{F} \tilde{P}_1^{1/2}] \\ &\geq \kappa(\tilde{F}),\end{aligned}$$

since the eigenvalues of $\tilde{F}$ are preserved under the nonsingular transformation. Assertion (b) follows immediately, i.e.,

$$\begin{aligned}\tilde{A} \in \mathrm{H}^- &\Leftrightarrow \mu[\tilde{A}] < 0 \Leftrightarrow \mu(T^T TA) < 0 \\ &\Leftrightarrow \mu(X) < 0 \Rightarrow \tilde{F} \in \mathrm{H} \Rightarrow \tilde{A} \in \tilde{\mathrm{H}}.\end{aligned}$$

Finally, $\tilde{F}$ is unitary by definition and hence normal matrix, or

$$\tilde{A} \in \tilde{\mathrm{H}} \Leftrightarrow 0 > \kappa(\tilde{F}) = \mu(\tilde{F}) = \mu(\tilde{A}\tilde{P}_1^{-1}) \Leftrightarrow \mu(SA) < 0,$$

which proves assertion (c).

Corollary 1: For any T, such that $\bar{P} = T^T T$ is an ELM for A, $S = T^T \tilde{P}_1 T$ is also an ELM for A.

Proof: It follows from assertions (b) and (c) in Theorem 1. Let $T^T T$ be an ELM for A i.e.,

$$\begin{aligned}\mu(T^T TA) < 0 &\Leftrightarrow \mu(\tilde{A}) < 0 \Leftrightarrow \tilde{A} \in \mathrm{H}^- \\ &\Rightarrow \tilde{A} \in \tilde{\mathrm{H}} \Leftrightarrow \mu(SA) < 0.\end{aligned}$$

Comments: The approach suggested in [1] for getting upper bounds for the solution P in (1) is always theoretically applicable since the symmetric part of the transformed matrix $\tilde{A}$ is negative definite. Corollary 1 illustrates the important fact that the s.v.d. approach does not introduce any conservatism concerning restrictions on the coefficient matrix for bounds validity. If $T^T T$ is an ELM for A, then the upper matrix bound in (5) becomes

$$P \leq \eta S, \quad \eta = \frac{1}{2}\lambda_1\{-Q[(SA)_s]^{-1}\}.$$

Since the main purpose is to get ILM, let $T = I$, i.e., $\tilde{A} \equiv A$, $\tilde{F} \equiv F$, $S = \tilde{P}_1 \equiv P_1$. Then, the assertions of Theorem 1 become:

Corollary 2: For any matrix $A \in \mathrm{H}$ one has

(a) $\lambda_1(A_s P_1^{-1}) \geq \kappa(F)$,

(b) $A \in \mathrm{H}^- \Rightarrow A \in \tilde{\mathrm{H}}$,

(c) $A \in \tilde{\mathrm{H}} \Leftrightarrow \mu(P_1 A) < 0 \Leftrightarrow \mu(P_2^{-1} A) < 0$.

In other words, $A \in \tilde{\mathrm{H}}$, if and only if, P_1 and P_2^{-1} are ILMs for A.

Denote $S_1 = (P_1 A)_s$ and $S_2 = (P_2^{-1} A)_s$.

Corollary 3: Let $A \in \tilde{\mathrm{H}}$. The solution P in (1) has the following upper matrix bounds

$$\begin{aligned}&P \leq \mu_1 P_1, \quad \mu_1 = \frac{1}{2}\lambda_1(-QS_1^{-1}), \\ &P \leq \mu_2 P_2^{-1}, \quad \mu_2 = \frac{1}{2}\lambda_1(-QS_2^{-1}).\end{aligned} \tag{7}$$

Proof: It follows from Corollary 2, assertion (c) and the matrix bound (5) applied for $T^T T = P_1$, $\eta = \mu_1$ and $T^T T = P_2^{-1}, \eta = \mu_2$, respectively.

Lemma 1: If $A \in \tilde{\mathrm{H}}$, the maximum eigenvalue and the trace of P in (1) have the following upper bounds:

$$\lambda_1(P) \leq l_1 = \min[\mu_1 \sigma_1(A), \mu_2 \sigma_n^{-1}(A)], \tag{8}$$

$$\begin{aligned}&\mathrm{tr}\,(P) \leq t_1 = \min[t_1', t_2', t_1'', t_2''], \\ &t_1' = \frac{1}{2}\lambda_1(-QS_1^{-1})\,\mathrm{tr}\,(P_1), \\ &t_2' = \frac{1}{2}\lambda_1(-QS_2^{-1})\,\mathrm{tr}\,(P_2^{-1}), \\ &t_1'' = -\frac{1}{2}\lambda_1^{-1}(S_1 P_1^{-2})\mathrm{tr}(QP_1^{-1}), \\ &t_2'' = -\frac{1}{2}\lambda_1^{-1}(S_2 P_2^2)\mathrm{tr}(QP_2).\end{aligned} \tag{9}$$

Proof: Bounds l_1 and $t_i', i = 1,2$, are obtained from the respective matrix bounds (7) for P.

Having in mind the s.v.d. of $A = U\Sigma V^T = FP_1 = P_2 F$, $F = UV^T$, the CALE (1) can be rewritten as

$$\begin{aligned}&Q = -P_1 F^T P - PFP_1 \Rightarrow QP_1^{-1} = -P_1 F^T PP_1^{-1} - PF, \\ &Q = -F^T P_2 P - PP_2 F \Rightarrow QP_2 = -F^T P_2 PP_2 - PP_2 FP_2.\end{aligned}$$

Application of the tr operator to both sides of the above equalities results in

$$\begin{aligned}\mathrm{tr}\,(QP_1^{-1}) &= -2\,\mathrm{tr}\,(PF) \\ &= -2\,\mathrm{tr}\,(P^{1/2} F_s P^{1/2}) \geq -2\mu(F)\,\mathrm{tr}\,(P),\end{aligned}$$

$$\operatorname{tr}(QP_2) = -2\operatorname{tr}(PP_2FP_2)$$
$$= -2\operatorname{tr}(P^{1/2}P_2F_sP_2P^{1/2}) \geq -2\mu(P_2FP_2)\operatorname{tr}(P).$$

Since $F_S = P_1^{-1}S_1P_1^{-1} = S_2 < 0$, bounds $t''_{1,2}$ are proved.

The upper trace bound t''_1 was proposed in [7].

The requirement $A \in \tilde{H}$ is less restrictive in comparison with the assumption that $A \in H^-$, due to the fact that $H^- \subseteq \tilde{H}$. Therefore, bounds (7), (8), and (9) presented here are less conservative with respect to the validity restrictions imposed on matrix A by the existing estimation approaches

Comments: The derived bounds (7), (8), and (9) are based on the s.v.d. of the coefficient matrix A and in this sense they differ from all available bounds. Nevertheless, the only specific procedure consists in getting A in the form $A = U\Sigma V^T$. The computational complexity introduced by the s.v.d. is completely comparable with that required by the existing approaches, all the more if A is a real matrix, then U and V are also real [2]. Once the decomposition is done, one can easily compute the matrices involved in the respective bounds as follows:

$$P_1 = V\Sigma V^T, P_1^{-1} = V\Sigma^{-1}V^T, P_1^{-2} = V\Sigma^{-2}V^T,$$
$$P_2 = U\Sigma U^T, P_2^{-1} = U\Sigma^{-1}U^T, P_2^2 = AA^T, F = UV^T,$$
$$S_1 = (P_1A)_s = V\Sigma V^T(UV^T + VU^T)V\Sigma V^T$$
$$= V\Sigma V^T F_s V\Sigma V^T, S_2 = (P_2^{-1}A)_s = F_s.$$

Matrix F is a normal one and hence it is unitarily similar to a diagonal matrix Λ i.e., $F = W\Lambda W^T$. If Λ is Hurwitz, the proposed bounds are all valid. Then, the inverses of F_s, S_1 and S_2 are obtained by inversing diagonal matrices, i.e.,

$$F_s = 2W\Lambda_{\mathrm{Re}}W^T,\ F_s^{-1} = \frac{1}{2}W\Lambda_{\mathrm{Re}}^{-1}W^T,$$

$$S_1^{-1} = \frac{1}{2}V\Sigma^{-1}V^TW\Lambda_{\mathrm{Re}}^{-1}W^TV\Sigma^{-1}V^{-1},\ S_2^{-1} = F_s^{-1},$$

where Λ_{Re} denotes a diagonal matrix containing the real parts of the eigenvalues of F, while (2) and (5) require the computation of a possibly ill-conditioned general matrix. Since F is a normal matrix, the computation of W and Λ should not be a problem. In any case, this computation is easier than the computation of the eigenvalues of A (required by necessity in order to guarantee positive definite solution P), which must be put in the form $A = C^TTC$, with C (unitary) and T (triangular) being complex matrices in the general case. Note also, that computation of the spectrum of A is not required at all if the s.v.d. is performed due to the following reasons. Matrix F must be Hurwitz in order to get valid bounds. If F is Hurwitz, this leads to

$$A = FP_1 \Rightarrow P_1A = P_1FP_1 \Rightarrow A^TP_1 + P_1A = P_1F_sP_1 < 0,$$

which is possible only if A is Hurwitz. Besides the fact that the approach proposed in [1] requires solution of a linear matrix inequality (LMI) and therefore the computational burden may be comparable with the one needed for the solution of the CALE, it demonstrates one essential drawback. It is not a priori known whether the ELM T^TT in (5) will provide satisfactory bound and it is quite natural to expect that one should solve a series of LMIs.

So far, the existing upper bounds are expressed entirely in terms of the eigenvalues of the symmetric part of the coefficient matrix. The proposed bounds also provide some new theoretical insight concerning in general the estimation problem for the CALE. It becomes clear that the singular values of A and its unitary part F play an important role in estimating the solution P.

A natural question, concerning possibilities for further extension of the set of stable matrices for which ILM exist, arises. The next Lemma provides an answer.

Lemma 2: Denote

$$A_i = R_iAR_i^{-1},\ R_{i+1}^2 = R_i(A_i^TA_i)^{1/2}R_i,\ i \in I_+,$$

where I_+ is the set of all positive integers. Let $R_1 = I$. Define the sets $H_i \equiv \{A : A_i \in \tilde{H}\}, i \in I_+$. For any $i \in I_+$, one has

$$H^- \subseteq H_i \subseteq H_{i+1}.$$

Proof: For $i = 1, R_1 = I, A_1 = A, H_1 \equiv \tilde{H}$, or $H^- \subseteq H_1$ in accordance with Corollary 2. Let for some $i \in I_+, i \neq 1$, one has $A \in H_i$, i.e. $R_iAR_i^{-1} \in \tilde{H}$. Due to Corollary 2, assertion (c), this is equivalent to

$$(A_i^TA_i)^{1/2}A_i \in H^- \Leftrightarrow R_i(A_i^TA_i)^{1/2}R_iA$$
$$= R_{i+1}^2A \in H^- \Leftrightarrow A_{i+1} \in H^-.$$

But, $H^- \subseteq \tilde{H} \Rightarrow A_{i+1} \in \tilde{H} \Rightarrow A \in H_{i+1} \Rightarrow H_i \subseteq H_{i+1}$.

Corollary 4:

(a) $A \in H_i \Leftrightarrow \mu(R_{i+1}^2A) < 0$,

(b) If $A \in H_i$ for some $i = i^* \in I_+$, then $A \in H_j$ and R_j^2 is an ILM for A for all $j > i^*, j \in I_+$.

This result shows that the set of Hurwitz matrices for which there exist valid upper bounds for the solution of the CALE can be further extended.

4. EXAMPLES

The applicability of the proposed bounds for P is illustrated by several numerical examples. The obtained upper scalar bounds (8) and (9), are compared with the best known ones (2) and (3), when these are valid.

Example 1: Consider the unitary matrices

$$U=\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix},\ V=\frac{1}{\sqrt{1+a^2}}\begin{bmatrix}1 & a\\ -a & 1\end{bmatrix},$$

where $a > 0$ and the positive diagonal matrix $\Sigma = \mathrm{diag}(\sigma,1)$. It is desired to investigate the influence of the parameters a and σ on the possibility to get bounds for P in (1), if the coefficient matrix is $A=U\Sigma V^T$. Denote $t=\left[\sqrt{2(1+a^2)}\right]^{-1}$. Then,

$$A=t\begin{bmatrix}-\sigma+a & 1+\sigma a\\ -\sigma-a & -1+\sigma a\end{bmatrix}.$$

For any $a < 1$ and any $\sigma>0$, $A\in \mathrm{H}$. The unitary part of A is

$$F=UV^T=t\begin{bmatrix}-1+a & 1+a\\ -(1+a) & -1+a\end{bmatrix}\Rightarrow F_s=t(-1+a)I,$$

or $A\in\mathrm{H}\Leftrightarrow A\in\tilde{\mathrm{H}}$ in this case. If $A\in\mathrm{H}^-$, then $a<\min(\sigma,\sigma^{-1})$, by necessity.

Consider the matrix bounds (7), with $S_1=t(-1+a)P_1^2$, $S_2=t(-1+a)I$,

$$P_1=V\Sigma V^T=2t^2\begin{bmatrix}\sigma+a^2 & -a\sigma+a\\ -a\sigma+a & a^2\sigma+1\end{bmatrix},$$

$$P_2^{-1}=U\Sigma^{-1}U^T=0.5\begin{bmatrix}\sigma^{-1}+1 & \sigma^{-1}-1\\ \sigma^{-1}-1 & \sigma^{-1}+1\end{bmatrix}.$$

Case 1: Let a = 0.5 and $\sigma=2$. Since $\mu(A)>0$, bounds (2) and (3) are not valid. Let $Q = I$ in (1). The exact solution P is

$$P=0.25t^{-1}\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix},\mathrm{tr}(P)=1.5t^{-1},\lambda_1(P)=t^{-1}.$$

Since $A\in\tilde{\mathrm{H}}$, bounds (8) and (9) are valid. In this case, $\mu_1=\mu_2=t^{-1}$. The matrix bound $\mu_2P_2^{-1}=t^{-1}(AA^T)^{-1/2}\equiv P$, which is evident from the fact that

$$A^TP_2^{-1}+P_2^{-1}A=-tI\Rightarrow t^{-1}P_2^{-1}=P,$$

since P is unique for any given matrices A and Q. Consider bounds (8) and (9). The maximum eigenvalue upper bound is

$$l_1=\min(2t^{-1},t^{-1})=t^{-1}=\lambda_1(P)$$

and the trace upper bound is

$$t_1=\min(3t^{-1},1.5t^{-1},1.5t^{-1},1.5t^{-1})=1.5t^{-1}=\mathrm{tr}(P).$$

The proposed upper scalar bounds coincide with the exact respective solution parameters in this case.

Case 2: Let $a = 0.5, \sigma=1.5$ and $Q = I$. In this case $\mu(A)=\lambda_1(A_s)=-0.125(5-\sqrt{10})t<0$ and $\lambda_2(A_s)=-0.125(5+\sqrt{10})t$, or $A\in\mathrm{H}^-$ and bounds (2) and (3) are valid. The exact solution P is

$$P=(6t)^{-1}\begin{bmatrix}5 & -1\\ -1 & 5\end{bmatrix},\ \mathrm{tr}(P)=(5/3)t^{-1},\ \lambda_1(P)=t^{-1}.$$

The upper bounds (2) and (3) are computed as

$$l_0=4(5-\sqrt{10})^{-1}t^{-1},\quad t_0=(8/3)t^{-1}.$$

The proposed bounds (8) and (9) are:

$$l_1=\min(1.5t^{-1},t^{-1})=t^{-1}=\lambda_1(P)<l_0$$
$$t_1=(5/3)t^{-1}=\mathrm{tr}(P)<t_0.$$

Example 2: Consider matrix

$$A=\begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & y\\ 1 & x & a\end{bmatrix},$$

which is stable for $a<\min[1,-(1+xy)]$ and $A\in\mathrm{H}^-$ for $a<-1-0.25(x+y)^2$.

Let Q in (1) be chosen as $Q=\mathrm{diag}(3,2,1)$.

Case 1: Let a = -1, x = 1, and y = -2. In this case $\mu(A)>0$, and bounds (2) and (3) are not valid, while $A\in\tilde{\mathrm{H}}$ and the proposed bounds (8) and (9) can be used. The maximum eigenvalue and trace of P are computed as $\lambda_1(P)=4.2826$ and tr(P) = 5.95. The scalar upper bounds (8) and (9) are

$$l_1=\min(28.1978,5.2086)=5.2086,$$
$$t_1=7.4593.$$

Case 2: Let a = -2, x = 1, and y = -2. Since $\mu(A)<0\Rightarrow A\in\mathrm{H}^-\Rightarrow A\in\tilde{\mathrm{H}}$, bounds (2), (3), (8), and (9) are valid. In this case $\lambda_1(P)=2.7345$ and tr(P) = 4. The respective scalar bounds are:

$$l_0=4.2106,\quad t_0=6.633,$$

$$l_1 = \min(15.2162, 2.8246) = 2.8246 < l_0,$$
$$t_1 = 4.407 < t_0.$$

Example 3: Consider matrix

$$A = \begin{bmatrix} -2 & 1 & 0 & 5 \\ -3 & -2 & 2 & 0 \\ 0 & -3 & -5 & 0 \\ -4 & 0 & 3 & a \end{bmatrix}.$$

et $Q = (A^T A)^{1/2}$.

Case 1: Let a = -0.6. Then $A \in \tilde{\mathrm{H}}$, but $\mu(A) > 0$ nd bounds (2) and (3) are not valid. The computed naximum eigenvalue and trace of P in (1) and their espective bounds (8) and (9) are

$$\lambda_1(P) = 2.3154, \quad \mathrm{tr}(P) = 4.9906,$$
$$l_1 = \min(7.494, 10.7412) = 7.494,$$
$$t_1 = 8.6243.$$

Case 2: Let a = -2. In this case $\mu(A) < 0$ and

$$\lambda_1(P) = 1.4742, \quad \mathrm{tr}(P) = 3.803,$$
$$l_0 = 1.9103, \quad t_0 = 7.9844,$$
$$l_1 = \min(1.8885, 6.3628) = 1.8885 < l_0,$$
$$t_1 = 4.9454 < t_0.$$

Define the percentage error in estimating the maximum eigenvalue and trace of P, respectively, as:

$$\Delta_{l_i}[\%] = [(l_i / \lambda_1(P) - 1] \times 100, \quad i = 0,1.$$
$$\Delta_{t_i}[\%] = [(t_i / \mathrm{tr}(P) - 1] \times 100, \quad i = 0,1.$$

The results obtained from the above examples can be summarized as follows:

Example 1:

Case 1: $\Delta_{l_1} = \Delta_{t_1} = 0$; bounds (2) and (3) are not valid.

Case 2:

$$\Delta_{l_1} = 0 < \Delta_{l_0} = 100.42\%; \Delta_{t_1} = 0 < \Delta_{t_0} = 60\%.$$

Example 2:

Case 1: $\Delta_{l_1} = 21.62\%$, $\Delta_{t_1} = 25.37\%$; bounds (2) and (3) are not valid.

Case 2:

$$\Delta_{l_1} = 3.295\% < \Delta_{l_0} = 54\%,$$
$$\Delta_{t_1} = 10.175\% < \Delta_{t_0} = 65.8\%.$$

Example 3:

Case 1: $\Delta_{l_1} = 223.66\%$, $\Delta_{t_1} = 72.8\%$; bounds (2) and (3) are not valid.

Case 2:

$$\Delta_{l_1} = 28.1\% < \Delta_{l_0} = 29.6\%,$$
$$\Delta_{t_1} = 30\% < \Delta_{t_0} = 110\%.$$

The best known upper bounds (2) and (3) cannot be used in half of the considered cases. In addition, the proposed bounds (8) and (9) are much tighter then bounds (2) and (3) in the cases when the last ones are valid.

5. CONCLUSION

Extending the set of Hurwitz matrices for which upper solution bounds for the CALE are valid is the topic of this paper. The available bounds require that $A \in \mathrm{H}^-$ (set of matrices with negative definite symmetric parts), which is a very restrictive condition. Applying the s.v.d. approach, any nonsingular matrix can be represented as a product of a unitary matrix F and a positive definite matrix. It is proved in Theorem 1 that:

(i) $\mathrm{H}^- \subseteq \tilde{\mathrm{H}}$ (set of matrices with stable unitary parts F),

(ii) $A \in \tilde{\mathrm{H}} \Leftrightarrow P_1 = (A^T A)^{1/2}$ and $P_2^{-1} = (AA^T)^{-1/2}$ are ILMs for A.

This helps to achieve the proposed upper matrix and scalar solution bounds that are proposed here (Corollary 3 and Lemma 1). A further extension of the set $\tilde{\mathrm{H}}$ is defined in Lemma 2.

The main contribution of this work consists in the attempt made to overcome some well known difficulties concerning solution bounds for the CALE, such as hard validity restrictions and additional computational burden when the ELM approach to get estimates is used. Several examples illustrate the validity and superiority of the new bounds over the best available ones.

REFERENCES

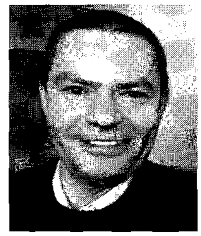
[1] Y. Fang, K. Loparo, and X. Feng, "New estimates for solutions of Lyapunov equations," *IEEE Trans. on Automatic Control*, vol. 42, no. 3, pp. 408-411, March 1997.

[2] R. Horn and C. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[3] N. Komaroff, "Upper summation and product bounds for solution eigenvalues of the Lyapunov matrix equation," *IEEE Trans. on Automatic Control*, vol. 37, no. 5, pp. 337-341, May 1992.

[4] W. Kwon, Y. Moon, and S. Ahn, "Bounds in algebraic Riccati and Lyapunov equations: A survey and some new results," *International Journal of Control*, vol. 64, no. 3, pp. 377-389,

March 1996.
[5] T. Mori and I. Derese, "A brief summary of the bounds on the solution of the algebraic matrix equations in control theory," *International Journal of Control,* vol. 39, no. 2, pp. 247-256, February 1984.
[6] T. Mori and H. Kokame, "On solution bounds for three types of Luapunov matrix equations: Continuous, discrete and unified equations," *IEEE Trans. on Automatic Control*, vol. 47, no. 10, pp. 1767-1770, October 2002.
[7] S. Savov and I. Popchev, "New upper estimates for the solution of the CALE," *IEEE Trans. on Automatic Control,* vol. 49, no. 10, pp. 1841-1842, October 2004.
[8] M. Tippett and D. Marchesin, "Upper bounds for the solution of the discrete algebraic Lyapunov equation," *Automatica*, vol. 35, no. 8, pp. 1485-1489, 1999.
[9] K. Yasuda and K. Hirai, "Upper and lower bounds on the solution of the algebraic matrix Riccati and Lyapunov equation," *IEEE Trans. on Automatic Control*, vol. 24, no. 6, pp. 483-487, June 1979.



Svetoslav G. Savov received the B.S. degree in Electrical Engineering from the Technical University, Sofia, in 1982. His research interests include robust stability for polytopic systems and solution estimates for Lyapunov equations.

Ivan P. Popchev received the B.S. degree in Electrical Engineering from the Technical University, Sofia, in 1959. His research interests include control theory and mathematical modeling in economics.