Fuzzy Optim Decis Making (2007) 6:351–365
DOI 10.1007/s10700-007-9018-6

Aggregation of fuzzy preference relations to multicriteria decision making

Vania Peneva · Ivan Popchev

Published online: 24 October 2007

Abstract Weighted aggregation of fuzzy preference relations on the set of alternatives by several criteria in decision-making problems is considered. Pairwise comparisons with respect to importance of the criteria are given in fuzzy preference relation as well. The aggregation procedure uses the composition between each two relations of the alternatives. The membership function of the newly constructed fuzzy preference relation includes t-norms and t-conorms to take into account the relation between the criteria importance. Properties of the composition and new relation, giving a possibility to make a consistent choice or to rank the alternatives, are proved. An illustrative numerical study and comparative examples are presented.

Keywords Fuzzy preference relations · Composition of fuzzy relations · Transitivity properties · T-norms · T-conorms · Aggregation operators · Decision making

1 Introduction

Most multicriteria decision making models have been developed using mainly fuzzy preference relations. One of the problems of these models concerns an aggregation of such relations into a union relation with properties providing a possibility to make a consistent choice or to rank the alternatives from the "best" to the "worst" one. The concept of fuzzy majority represented by a linguistic quantifier to aggregate fuzzy preference relations is used in (Tanino 1984; Kacprzyk 1986; Chiclana et al. 1998; Ma et al. 1999). The aggregation operators are also being used here, refer e.g. to Weighted Mean (Chiclana et al. 1998; Yager 1994, 1996), Weighted Geometric (Chiclana et al. 2000), Weighted MaxMin and Weighted MinMax (Fodor and Roubens 1995; Roubens

V. Peneva (✉) · I. Popchev
Institute of Information Technologies, Acad. G. Bonchev Str., Bl. 2, 1113 Sofia, Bulgaria
e-mail: vgpeneva@yahoo.co.uk

1995) operators. Several operators with parameters are introduced, e.g. MaxMin, MinAvg, Gamma operators (Zimmermann 1993), Generalized Mean operator (Grabisch et al. 1998; Detyniecki 2001; Sousa and Kaymak 2001).

Very good overviews of the aggregation operators along with their advantages and disadvantages are provided by (Grabisch et al. 1998; Detyniecki 2001; Mesiar 2003).

Different approaches to aggregation are investigated in (Xu 2004a, 2005; Jin and Sendhoff 2002; Calvo and Mesiar 2003).

To make a consistent choice or rank of the alternatives from the "best" to the "worst" one, when fuzzy preference relations are assumed, a set of properties to be satisfied have been suggested. The consistency in this case has a direct effect on the ranking results of the final decision. The investigations on the consistency of fuzzy preference relations are made in (Xu and Da 2003; Xu 2004b; Herrera-Viedma et al. 2004; Ma et al. 2006). Studying consistency is associated with the concept of transitivity (Basile 1990; Herrera-Viedma et al. 2004). Let $\mu : A \times A \to [0, 1]$ be a membership function of a fuzzy relation and $a, b, c \in A$. Some definitions for transitivity are:

- Max–min transitivity (Dubois and Prade 1980; Zimmermann 1993): $\mu(a, c) \geq \min(\mu(a, b), \mu(b, c))$;
- Max–max transitivity (Tanino 1988): $\mu(a, c) \geq \max(\mu(a, b), \mu(b, c))$;
- Restricted max–min transitivity (Tanino 1988): $\mu(a, b) \geq 0.5,\ \mu(b, c) \geq 0.5 \Rightarrow \mu(a, c) \geq \min(\mu(a, b), \mu(b, c))$;
- Restricted max–max transitivity (Tanino 1988): $\mu(a, b) \geq 0.5,\ \mu(b, c) \geq 0.5 \Rightarrow \mu(a, c) \geq \max(\mu(a, b), \mu(b, c))$;
- Additive transitivity (Tanino 1984, 1988): $\mu(a, c) = \mu(a, b) + \mu(b, c) - 0.5$.

Characterizations and comparisons between these transitivity are suggested in (Herrera-Viedma et al. 2004). The additive transitivity is a stronger property than restricted max–max one, which is a stronger concept than the restricted max–min transitivity, but a weaker property than max–max transitivity. The latter property is a stronger one than max–min transitivity, which is a stronger property in comparison with the restricted max–min transitivity. Methods for constructing fuzzy preference relations from preference data are described in (Ekel 2002; Xu 2004a; Herrera-Viedma et al. 2004; Alonso et al. 2005). Applying these methods makes possible to get consistency of the fuzzy preference relations and thus to avoid inconsistent solutions in the decision making processes.

Zadeh (1971) suggested several useful definitions for transitivity, which are compared in (Venugopalan 1992). The weakest of them is the max-Δ transitivity, i.e. $\mu(a, c) \geq \max(0, \mu(a, b) + \mu(b, c) - 1)$. It is shown that this is the most suitable notion of transitivity for fuzzy ordering.

The dependence between the properties of the aggregated relation and the individual relations is investigated in (Chiclana et al. 2003, 2004; Drewniak and Dudziak 2004; Jacas and Recasens 2003; Mesiar and Saminger 2004; Roubens 1989; Peneva and Popchev 2003, 2005).

The fuzzy preference relations are the basic concept in the following multicriteria decision making problem considered here. Let $A = \{a_1, \ldots, a_n\}$ be the finite set of alternatives evaluated by several fuzzy criteria $K = \{k_1, \ldots, k_m\}$, i.e. these criteria give fuzzy preference relations $R_1, R_2, \ldots, R_m$ between the alternatives. When the cardinality n of A is small, the preference relations may be represented by the $n \times n$ matrices $R_k = ||r_{ij}^k||$, where $r_{ij}^k = \mu_k(a_i, a_j)$, $i, j = 1, \ldots, n$, $k = 1, \ldots, m$, μ_k: $A \times A \rightarrow [0, 1]$ is the membership function of the relation R_k and r_{ij}^k is the preference degree of the alternative a_i over a_j by the criterion k_k. $r_{ij}^k = 0.5$ indicates indifference between a_i and a_j, $r_{ij}^k = 1$ indicates that a_i is absolutely preferred to a_j, and $r_{ij}^k > 0.5$ indicates that a_i is preferred to a_j by the k-th criterion. In this case, the preference matrices R_k, $k = 1, \ldots, m$ are usually assumed to be additive reciprocal, i.e.

$$r_{ij}^k + r_{ji}^k = 1, \quad i, j = 1, \ldots, n.$$

A fuzzy preference relation W between the criteria is given as well, i.e. the couples of criteria are compared according to their importance. Let $W = ||w(k_i, k_j)|| = ||w_{ij}||$, $i, j = 1, \ldots, m$, where $w(k_i, k_j)$ be the preference degree of the criterion k_i over k_j. The general procedure to include these preference degrees in the aggregation process uses a transformation of the preference values r_{ij}^k under the importance degree w_{ij} to generate a new value. This transformation can be made with the help of a function with required properties. Examples of the transformation function include: the minimum operator (Yager 1994), an exponential function (Yager 1978), any t-norm operator (Zimmermann 1993; Peneva and Popchev 2005), a linguistic quantifier (Chiclana et al. 2001).

The setting problem is to obtain the preference relation between the alternatives uniting the fuzzy relations by the individual criteria taking into account the relation between the importance of the criteria. The aim is to use the whole information given above up to the final stage of the problem solution without transforming the relation W into some weighted coefficients.

2 Properties of a composition of fuzzy relations

An attempt to use the composition of two relations in an aggregation procedure is made here. If the composition possesses some properties required for the solution of ranking or choice problems, then it may be used in such procedures. This will give one practical application of the composition.

Definition 1 (Nguyen and Walker 2000). Let X and Y be fuzzy relations in $A \times A$ and let T be a t-norm. The composition $X \circ Y$ of these relations with respect to T is the fuzzy relation on $A \times A$ with membership function

$$\mu(a_i, a_j) = \mu_{X \circ Y}(a_i, a_j) = \max_k \{T(\mu_X(a_i, a_k), \mu_Y(a_k, a_j))\},$$
$$i, j, k = 1, \ldots, n. \quad (1)$$

When $T = \min$ then the composition is a max–min one. When $T = xy$, then it is a max-product composition. $X \circ Y$ can be obtained as the matrix product of X and Y. It has to be taken into account that $X \circ Y \neq Y \circ X$ in general.

Let $X = ||x_{ij}||$ and $Y = \|y_{ij}\|$, $i, j = 1, \ldots, n$ be fuzzy relations in $A \times A$, where x_{ij}, y_{ij} are the membership degrees of the comparison of the alternatives $a_i, a_j \in A$ to the fuzzy preference relations X and Y, respectively.

Certainly, the properties of the composition of two relations depend on the relations' properties. Therefore, it needs to investigate what kind of transitivity must possess both relations in order their composition has some transitivity properties. The examples show that the composition does not preserve the transitivity property, but it transforms the additive and max–max transitivity into the max-Δ one (see Propositions 1, 2). Besides the composition of two restricted max–min or max–max transitivity relations is not always a max-Δ transitive relation.

Proposition 1 *If two fuzzy preference relations are additively transitive, then the composition of these relations is max-Δ transitive.*

Proof Let $X = ||x_{ij}||$ and $Y = ||y_{ij}||$, $i, j = 1, \ldots, n$ be additively transitive relations, i.e.

$$x_{ij} = x_{ik} + x_{kj} - 0.5, \quad x_{ij} + x_{ji} = 1 \quad \text{and} \quad y_{ij} = y_{ik} + y_{kj} - 0.5, \qquad (2)$$
$$y_{ij} + y_{ji} = 1, \ \forall i, j, k.$$

Two cases will be considered: max–min composition and max–product composition.
(A) *Max–min composition*

In this case if $Z = X \circ Y$, then in accordance with (1)

$$Z = \|z_{ij}\|, \quad z_{ij} = \max_k\{\min(x_{ik}, y_{kj})\}, \quad k = 1, \ldots, n. \qquad (3)$$

It has to be proved that

$$z_{ij} \geq \max(0, z_{ik} + z_{kj} - 1), \quad k = 1, \ldots, n. \qquad (4)$$

If $z_{ik} + z_{kj} - 1 \leq 0$, then (4) is proved, because $z_{ij} \in [0, 1]$.
Let $z_{ik} + z_{kj} - 1 > 0$, then in accordance with (3) let

$$z_{ik} = \max_s\{\min(x_{is}, y_{sk})\} = \min(x_{ik_1}, y_{k_1k}) = x_{ik_1}, \quad s = 1, \ldots, k_1, \ldots, n, \quad (5)$$

$$z_{kj} = \max_s\{\min(x_{ks}, y_{sj})\} = \min(x_{kk_2}, y_{k_2j}) = x_{kk_2}, \quad s = 1, \ldots, k_2, \ldots, n, \quad (6)$$

$$z_{ij} = \max_s\{\min(x_{is}, y_{sj})\} = \min(x_{ik_3}, y_{k_3j}) = x_{ik_3}, \quad s = 1, \ldots, k_3, \ldots, n. \quad (7)$$

If $x_{ik_3} \geq x_{ik_1}$ or $x_{ik_3} \geq x_{kk_2}$, then (4) is proved. Let $x_{ik_3} < x_{ik_1}$ and $x_{ik_3} < x_{kk_2}$. From (7) one has $\min(x_{ik_3}, y_{k_3j}) \geq \min(x_{ik_1}, y_{k_1j})$, therefore $x_{ik_3} \geq \min(x_{ik_1}, y_{k_1j})$

$= y_{k_1j}$ and it has to be proved that $x_{ik_3} \geq y_{k_1j} \geq \min(x_{ik_1}, y_{k_1k}) + \min(x_{kk_2}, y_{k_2j}) - 1$, but $\min(x_{ik_1}, y_{k_1k}) + \min(x_{kk_2}, y_{k_2j}) - 1 \leq y_{k_1k} + y_{k_2j} - 1$. Hence, if

$$y_{k_1j} \geq y_{k_1k} + y_{k_2j} - 1, \tag{8}$$

then (4) is valid. Taking into account that the relation Y is reciprocal, inequality (8) becomes

$$y_{k_1j} \geq y_{k_1k} - y_{jk_2} \Leftrightarrow y_{k_1j} + y_{jk_2} \geq y_{k_1k}. \tag{9}$$

Relation Y is additively transitive and therefore $y_{k_1k_2} = y_{k_1j} + y_{jk_2} - 0.5$, i.e. $y_{k_1j} + y_{jk_2} = y_{k_1k_2} + 0.5$, then (9) becomes

$$y_{k_1k_2} + 0.5 \geq y_{k_1k} = 1 - y_{kk_1} \Leftrightarrow y_{kk_1} + y_{k_1k_2} - 0.5 \geq 0 \Leftrightarrow y_{kk_2} \geq 0.$$

The proofs for other variants of minimum values in (5), (6), and (7) are reduced to the above case.

(B) *Max-product composition*

In this case if $Z = X \circ Y$, then according to (1)

$$Z = ||z_{ij}||, \quad z_{ij} = \max_k\{x_{ik} \,.\, y_{kj}\}, \quad k = 1, \ldots, n. \tag{10}$$

Using the above notations, it follows that

$$z_{ik} = \max_s\{x_{is} \,.\, y_{sk}\} = x_{ik_1} \cdot y_{k_1k}, \quad s = 1, \ldots, k_1, \ldots, n, \tag{11}$$

$$z_{kj} = \max_s\{x_{ks} \,.\, y_{sj}\} = x_{kk_2} \cdot y_{k_2j}, \quad s = 1, \ldots, k_2, \ldots, n, \tag{12}$$

$$z_{ij} = \max_s\{x_{is} \,.\, y_{sj}\} = x_{ik_3} \cdot y_{k_3j}, \quad s = 1, \ldots, k_3, \ldots, n. \tag{13}$$

Let $z_{ik} + z_{kj} - 1 > 0$ and $z_{ij} < z_{ik}$, $z_{ij} < z_{kj}$, then according to (4), (11), (12), and (13) it has to be proved that

$$x_{ik_3} \cdot y_{k_3j} \geq x_{ik_1} \cdot y_{k_1k} + x_{kk_2} \cdot y_{k_2j} - 1. \tag{14}$$

From (13) it follows that

$$z_{ij} = \max_s\{x_{is} \,.\, y_{sj}\} = x_{ik_3} \cdot y_{k_3j} \geq x_{ik_1} \cdot y_{k_1j}.$$

Then, if

$$x_{ik_1} \cdot y_{k_1j} \geq x_{ik_1} \cdot y_{k_1k} + x_{kk_2} \cdot y_{k_2j} - 1 \tag{15}$$

is valid, (14) will be valid, as well.

From $z_{ij} < z_{ik}$ it follows that $x_{ik_1} \cdot y_{k_1 j} < x_{ik_1} \cdot y_{k_1 k}$, i.e. $y_{k_1 j} < y_{k_1 k}$ and then (15) becomes

$$x_{ik_1}(y_{k_1 j} - y_{k_1 k}) \geq x_{kk_2} \cdot y_{k_2 j} - 1 \Leftrightarrow x_{ik_1}(y_{kk_1} + y_{k_1 j} - 0.5 - 0.5) \geq x_{kk_2} \cdot y_{k_2 j} - 1$$
$$\Leftrightarrow x_{ik_1}(y_{kj} - 0.5) \geq x_{kk_2} \cdot y_{k_2 j} - 1 \Leftrightarrow 1 - x_{kk_2} \cdot y_{k_2 j} \geq x_{ik_1}(0.5 - y_{kj}).$$

- If $x_{ik_1} \leq x_{kk_2}$ and $y_{kj} \leq 0.5 \rightarrow x_{ik_1}(0.5 - y_{kj}) \leq x_{kk_2}(0.5 - y_{kj})$. Then, it has to be proved that $1 - x_{kk_2} \cdot y_{k_2 j} \geq x_{kk_2}(0.5 - y_{kj})$, but

$$1 - x_{kk_2} \cdot y_{k_2 j} - x_{kk_2}(0.5 - y_{kj}) = 1 - x_{kk_2}(y_{k_2 j} + 0.5 - y_{kj})$$
$$= 1 - x_{kk_2}(y_{k_2 j} + 0.5 - 1 + y_{jk}) = 1 - x_{kk_2}(y_{k_2 j} + y_{jk} - 0.5) = 1 - x_{kk_2} y_{k_2 k} \geq 0.$$

- If $x_{ik_1} > x_{kk_2}$, then $1 - x_{kk_2} y_{k_2 j} > 1 - x_{ik_1} y_{k_2 j}$ and from $y_{kj} \leq 0.5$ it has to be proved that

$$1 - x_{ik_1} \cdot y_{k_2 j} \geq x_{ik_1}(0.5 - y_{kj}) \Leftrightarrow 1 - x_{ik_1} \cdot y_{k_2 j} - x_{ik_1}(0.5 - y_{kj})$$
$$= 1 - x_{ik_1}(y_{k_2 j} + 0.5 - y_{kj}) = 1 - x_{ik_1} y_{k_2 j} \geq 0.$$

The proof for the transitivity of the composition $Y \circ X$ is done in a similar way.

The additive transitivity does not imply the max–max one (e.g. see matrix X from Sect. 4), and due to this the following proposition is suggested.

Proposition 2 *If two fuzzy preference relations are max–max transitive, then the composition of these relations is max-Δ transitive.*

Proof Let $X = \|x_{ij}\|$ and $Y = \|y_{ij}\|$, $i, j = 1, \ldots, n$ be max–max transitive relations, i.e.

$$x_{ij} \geq \max(x_{ik}, x_{kj}) \quad \text{and} \quad y_{ij} \geq \max(y_{ik}, y_{kj}), \quad \forall i, j, k.$$

If $Z = X \circ Y$, $Z = \|z_{ij}\|$, it has to be proved that

$$z_{ij} \geq \max(0, z_{ik} + z_{kj} - 1), \quad k = 1, \ldots, n.$$

(A) *Max–min composition*

According to (5), (6), and (7), let $z_{ik} = x_{ik_1}$, $z_{kj} = x_{kk_2}$, $z_{ij} = x_{ik_3}$, i.e. it has to be proved that $x_{ik_3} \geq \max(0, x_{ik_1} + x_{kk_2 - 1})$.

But $x_{ik_3} \geq \max(x_{ik_1}, x_{k_1 k_3}) \geq x_{ik_1} \geq x_{ik_1} x_{kk_2} \geq \max(0, x_{ik_1} + x_{kk_2} - 1)$. The proofs for the other variants of minimum values in (5), (6), and (7) are reduced to this case.

(B) *Max–product composition*

According to (11), (12), and (13), let $z_{ik} = x_{ik_1} \cdot y_{k_1 k}$, $z_{ij} = x_{ik_3} \cdot y_{k_3 j}$, i.e. it has to be proved that $x_{ik_3} \cdot y_{k_3 j} \geq \max(0, x_{ik_1} \cdot y_{k_1 k} + x_{kk_2} \cdot y_{k_2 j} - 1)$. But for $\forall s$, $x_{ik_3} \geq \max(x_{is}, x_{sk_3})$, i.e. $x_{ik_3} \geq x_{ik_1}$ and $y_{k_3 j} \geq y_{k_2 j}$. Therefore

$$x_{ik_3} \cdot y_{k_3 j} \geq x_{ik_1} \cdot y_{k_2 j} \geq \max(0, x_{ik_1} + y_{k_2 j} - 1) \geq \max(0, x_{ik_1} \cdot y_{k_1 k} + x_{kk_2} \cdot y_{k_2 j} - 1).$$

The numerical example below shows that the composition of two fuzzy relations does not preserve the other properties of the fuzzy relations.

3 Approach for aggregation of fuzzy relations with different importance

Let $X = \|x_{ij}\|$ and $Y = \|y_{ij}\|$, $i, j = 1, \dots, n$ be fuzzy relations in $A \times A$, where x_{ij}, y_{ij} are the membership degrees of the comparison of the alternatives $a_i, a_j \in A$ to the fuzzy preference relations X and Y, respectively. Taking into account the relation W, a new fuzzy relation R between X and Y, $X \neq Y$, with the following membership degrees

$$r_{ij} = \begin{cases} 0.5 & \text{if } a_i = a_j \\ S(T(w^1, z^1_{ij}), T(w^2, z^2_{ij})) & \text{if } a_i \neq a_j \end{cases}, \tag{16}$$

is suggested, where

$$Z^1 = X \circ Y = \left\| z^1_{ij} \right\|, \quad z^1_{ij} = \max_k \{T(x_{ik}, y_{kj})\},$$
$$Z^2 = Y \circ X = \left\| z^2_{ij} \right\|, \quad z^2_{ij} = \max_k \{T(y_{ik}, x_{kj})\}, \quad k = 1, 2, \dots, n,$$

$w^1 = w(k_X, k_Y)$, $w^2 = w(k_Y, k_X)$ are the preference degrees of the criterion with a relation X over Y and Yover X, respectively, T is a t-norm and S is a corresponding t-conorm.

The main idea used in (16) consists in the fact that the composition of two relations compares the preference degrees of the i-th alternative with all alternatives from the first relation with the preference degrees of all alternatives to the j-th alternative from the second relation and vice versa, since the composition operation is not commutative. Then, taking into account the relation W, i.e. w^1 and w^2 are the preference degrees of relation X over Y and Yover X, respectively, a comparison operator that "pessimistically" combines the relations W and Z^1, W and Z^2 is used to get match measures which can be after that "optimistically" united in an overall result. Thus, if a t-norm provides the "pessimistic" combination, a t-conorm plays the role of an "optimistic" union. If the fuzzy relations corresponding to the criteria are $R_1, R_2, \dots, R_m$ and R_{ij}, $i \neq j$, is the new fuzzy relation according to (16) and since $R_{ij} = R_{ji}$, the number k of the new relations will be equal to the combinations of two elements over m, i.e., $k = \frac{m(m-1)}{1.2}$. Aggregation operators (Peneva and Popchev 2003) uniting these k relations can be used after that to obtain the aggregation fuzzy relation giving a possibility to decide the choice or ranking problems.

The purpose of the following investigation is to prove that the fuzzy relation with membership degrees (16) preserves, or does not preserve the transitivity property of the individual fuzzy relations.

Proposition 3 *If the relations Z^1, Z^2 are max-Δ transitive ones and the relation W is additive reciprocal, then the relation (16) is max-Δ transitive for the couple of t-norms* ($T = \min$, $S = \max$) *and* ($T = xy$, $S = x + y - xy$).

Proof Let Z^1, Z^2 be max-Δ transitive relations, i.e.

$$z_{ij}^1 \geq \max(0, z_{ik}^1 + z_{kj}^1 - 1), \quad z_{ij}^2 \geq \max(0, z_{ik}^2 + z_{kj}^2 - 1), \quad k = 1, \ldots, n \quad (17)$$

and W is additive reciprocal, i.e.

$$w^1 + w^2 = 1. \quad (18)$$

Then, it has to be proved that

$$S(T(w^1, z_{ij}^1), T(w^2, z_{ij}^2) \geq \\ \max(0, S(T(w^1, z_{ik}^1), T(w^2, z_{ik}^2)) + S(T(w^1, z_{kj}^1), T(w^2, z_{kj}^2)) - 1). \quad (19)$$

The following notations will be used for simplicity:

$$z_{ij}^1 = r \quad z_{ij}^2 = q \quad z_{ik}^1 = a \quad z_{ik}^2 = b$$
$$z_{kj}^1 = c \quad z_{kj}^2 = d \quad w^1 = x \quad w^2 = y$$

Then (17), (18), and (19) may be rewritten as:

$$r \geq \max(0, a + b - 1), \quad q \geq \max(0, c + d - 1), \quad x + y = 1, \quad (20)$$

$$S(T(x, r), T(y, q)) \geq \max(0, S(T(x, a), T(y, b)) \\ + S(T(x, c), T(y, d)) - 1). \quad (21)$$

If $S(T(x, r), T(y, q)) \geq S(T(x, a), T(y, b))$ or $S(T(x, r), T(y, q)) \geq S(T(x, c), T(y, d))$, or $S(T(x, a), T(y, b)) + S(T(x, c), T(y, d) - 1 \leq 0$, then (21) is valid. For the rest of the cases it has to be proved that

$$S(T(x, r), T(y, q)) \geq S(T(x, a), T(y, b)) + S(T(x, c), T(y, d)) - 1. \quad (22)$$

The most complicated case will be considered, taking into account (20). Let

$$0 \leq r \leq a \leq c \leq 1, \quad 0 \leq q \leq b \leq d \leq 1, \quad 0 \leq x \leq y \leq 1. \quad (23)$$

(A) Let $T = \min$, $S = \max$. In this case from (22) it has to be proved that

$$\max(\min(x, a), \min(y, b)) + \max(\min(x, c), \min(y, d)) - 1 \\ \leq \max(\min(x, r), \min(y, q)). \quad (24)$$

Taking into account (23), the position of x between r, a, $c$and the position of y between q, b, d, one faces the following cases:

- Let $x \leq r$, $y \leq q$, then (24) becomes $y + y - 1 \leq y$, which is valid.

- Let $r \leq x \leq a, y \leq q$, then (24) becomes $\max(x, y) + \max(x, y) - 1 \leq \max(r, y)$, but $r \leq x \leq y$ and this case reduces to 1.
- Let $a \leq x \leq c, y \leq q$, then from (24) follows $\max(a, y) + \max(x, y) - 1 \leq \max(r, y)$, but $a \leq x \leq y,\ r \leq x \leq y$ and this case reduces to 1.
- Let $c \leq x \leq 1, y \leq q$, then (24) becomes $\max(a, y) + \max(c, y) - 1 \leq \max(r, y)$, but $r \leq a \leq c \leq x \leq y$ and this case reduces to 1.
- Let $x \leq r,\ q \leq y \leq b$, then (24) becomes $\max(x, y) + \max(x, y) - 1 \leq \max(x, q)$,
  - if $\max(x, q) = x$, then it has to be proved that $y + y - 1 \leq x$, but according to (23) and (20) $y + y - 1 \leq b + d - 1 \leq q \leq x$,
  - if $\max(x, q) = q$, then (24) becomes $y + y - 1 \leq q$, that is valid.
- Let $x \leq r,\ b \leq y \leq d$, then (24) becomes $\max(x, b) + \max(x, y) - 1 \leq \max(x, q)$, but $\max(x, b) + \max(x, y) - 1 \leq \max(x, y) + \max(x, y) - 1 \leq \max(x, q)$ according to 5.
- Let $x \leq r,\ d \leq y \leq 1$, then (24) becomes $\max(x, b) + \max(x, d) - 1 \leq \max(x, q)$,
  - if $\max(x, q) = q$, then $x \leq q \leq b \leq d$ and (24) becomes $b + d - 1 \leq q$, that is valid according to (20), if $\max(x, q) = x$ and $x \leq b$, then (24) becomes $b + d - 1 \leq q \leq x$,
  - if $\max(x, q) = x$ and $b \leq x \leq d$, then (24) is $x + d - 1 \leq x$, that is valid,
  - if $\max(x, q) = x$ and $x > d$, then (24) is $x + x - 1 \leq x$, that is valid.
- Let $r \leq x \leq a,\ q \leq y \leq b$, then (24) becomes $y + y - 1 \leq \max(r, q)$,
  - if $\max(r, q) = q$, then according to (20) and (23) one has $y + y - 1 \leq b + d - 1 \leq q$,
  - if $\max(r, q) = r$, then $y + y - 1 \leq b + d - 1 \leq q \leq r$.
- Let $r \leq x \leq a,\ b \leq y \leq d$, then (24) becomes $\max(x, b) + \max(x, y) - 1 \leq \max(r, q)$,
  - if $x \leq b,\ r \leq q$, then one has $b + y - 1 \leq b + d - 1 \leq q$, that is valid according to (20),
  - if $x \leq b,\ r > q$, then $b + y - 1 \leq b + d - 1 \leq q \leq r$,
  - if $x > b,\ r \leq q$, then $x + y - 1 = 1 - 1 = 0 \leq q$,
  - if $x > b,\ r > q$, then (24) becomes $x + y - 1 = 1 - 1 = 0 \leq r$.
- Let $r \leq x \leq a,\ d \leq y \leq 1$ then (24) becomes $\max(x, b) + \max(x, d) - 1 \leq \max(r, q)$, but $\max(x, b) + \max(x, d) - 1 \leq \max(x, b) + \max(x, y) - 1 \leq \max(r, q)$ that is valid according to 9.
- Let $a \leq x \leq c,\ q \leq y \leq b \rightarrow a \leq x \leq y$, then (24) becomes $y + y - 1 \leq \max(r, q)$,
  - if $\max(r, q) = q$, then according to (20) and (23) one has $y + y - 1 \leq b + d - 1 \leq q$,
  - if $\max(r, q) = r$, then $y + y - 1 \leq b + d - 1 \leq q \leq r$.
- Let $a \leq x \leq c,\ b \leq y \leq d$, then (24) becomes $\max(a, b) + \max(x, y) - 1 \leq \max(r, q)$,

- if $a \leq b,\ r \leq q$, then one has according to (20) $b + y - 1 \leq b + d - 1 \leq q$,
- if $a \leq b,\ r > q$, then $b + y - 1 \leq b + d - 1 \leq q \leq r$,
- if $a > b,\ r \leq q$, then $x + y - 1 \leq 1 - 1 = 0 \leq q$,
- if $a > b,\ r > q$, then $x + y - 1 \leq r$.

- Let $a \leq x \leq c,\ d \leq y \leq 1$, then (24) becomes $\max(a, b) + \max(x, d) - 1 \leq \max(r, q)$, but $\max(a, b) + \max(x, d) - 1 \leq \max(a, b) + \max(x, y) - 1 \leq \max(r, q)$, that is valid according to 12.
- Let $c \leq x \leq 1,\ q \leq y \leq b$, then (24) becomes $\max(a, y) + \max(c, y) - 1 \leq \max(r, q)$, but $\max(a, y) + \max(c, y) - 1 \leq \max(a, b) + \max(x, y) - 1 \leq \max(r, q)$, that is valid according to 12.
- Let $c \leq x \leq 1,\ b \leq y \leq d \to c \leq x \leq y$, then (24) becomes $\max(a, b) + \max(c, y) - 1 \leq \max(r, q)$, but $\max(a, b) + \max(c, y) - 1 \leq \max(a, b) + \max(x, y) - 1 \leq \max(r, q)$, that is valid according to 12.
- Let $c \leq x \leq 1,\ d \leq y \leq 1$, then (24) becomes $\max(a, b) + \max(c, d) - 1 \leq \max(r, q)$, but $\max(a, b) + \max(c, d) \leq x + y \leq 1$ and hence the above inequality is valid.

Therefore, (19) is proved for $T = \min,\ S = \max$.
(B) Let $T = xy,\ S = x + y - xy$. In this case it has to be proved that (from (22))

$$xr + yq - xyrq \geq xa + yb - xyab + xc + yd - xycd - 1. \tag{25}$$

After simple transformations and using (20), (25) becomes

$$x(r - a - c + 1) + y(q - b - d + 1) + xy(ab + cd - rq) \geq 0. \tag{26}$$

Let $0 \leq r \leq a \leq c \leq 1,\quad 0 \leq q \leq b \leq d \leq 1$, then $rq \leq ab \leq cd$ and (25) is valid.
Let $0 \leq a \leq r \leq c \leq 1,\quad 0 \leq q \leq b \leq d \leq 1$, then $rq \leq cd$ and (25) is valid.
Let $0 \leq a \leq r \leq c \leq 1,\quad 0 \leq b \leq q \leq d \leq 1$, then $S(T(x, r), T(y, q)) \geq S(T(x, a), T(y, b))$ and (22) is valid. The other cases reduce to the last one.

Therefore, (19) is proved for these t-norms and corresponding t-conorms. The other t-conorms do not preserve the max-Δ transitivity of fuzzy relations as it is proved in (Peneva and Popchev 2005).

4 Numerical example

Let two criteria provide the following fuzzy preference relations on four alternatives.

$$X = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.55 & 0.7 & 0.95 \\ 0.45 & 0.5 & 0.65 & 0.9 \\ 0.3 & 0.35 & 0.5 & 0.75 \\ 0.05 & 0.1 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

$$
Y = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}
\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\
\begin{bmatrix} 0.5 & 0.3 & 0.2 & 0.3 \\ 0.7 & 0.5 & 0.4 & 0.5 \\ 0.8 & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.5 & 0.4 & 0.5 \end{bmatrix} \end{array}
$$

Suppose that the importance comparison of the two criteria are given by the following fuzzy preference relation

$$
W = \begin{array}{c} \\ w^1 \\ w^2 \end{array}
\begin{array}{c} \begin{array}{cc} w^1 & w^2 \end{array} \\
\begin{bmatrix} 0.5 & 0.6 \\ 0.4 & 0.5 \end{bmatrix} \end{array}
$$

The relations X and Y are additively transitive and reciprocal ones and the relation W is reciprocal. The new relations $Z^1 = X \circ Y$, $Z^2 = Y \circ X$ are computed as *max–min compositions*:

$$
Z^1 = \begin{bmatrix} 0.7 & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.6 & 0.5 & 0.6 \\ 0.7 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.4 & 0.5 \end{bmatrix}
\quad
Z^2 = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.55 & 0.7 & 0.7 \\ 0.5 & 0.55 & 0.7 & 0.8 \\ 0.5 & 0.55 & 0.7 & 0.7 \end{bmatrix}
$$

One can verify that these relations are not reflexive, symmetrical, reciprocal, but they are max-Δ transitive according to Proposition 1. The fuzzy relation uniting X and Y together with W using (16) is computed for t-norm $T=xy$ and t-conorm $S = x+y-xy$. The aggregated relation R is max-Δ transitive according to Proposition 3:

$$
R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}
\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\
\begin{bmatrix} 0.5 & 0.488 & 0.44 & 0.488 \\ 0.536 & 0.5 & 0.496 & 0.5392 \\ 0.536 & 0.454 & 0.5 & 0.524 \\ 0.44 & 0.454 & 0.4528 & 0.5 \end{bmatrix} \end{array}
$$

This fuzzy preference relation is a fuzzy preorder (Dubois and Prade 1980). Then, the perfect antisymmetry relation R' of R (Nakamura 1986) is computed, i.e. if

$$
R(a,b) \geq R(b,a) \rightarrow R'(a,b) = R(a,b) \vee R'(b,a) = 0.
$$

In this example

$$
R' = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}
\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\
\begin{bmatrix} 0.5 & 0.496 & 0.536 & 0.5392 \\ 0 & 0.5 & 0.536 & 0.524 \\ 0 & 0 & 0.5 & 0.488 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \end{array}
$$

The relation R' is a fuzzy partial ordering according to definition in Venugopalan (1992) and it is obvious that this relation is a fuzzy linear ordering (Dubois and Prade 1980), i.e.

$$a_2 \overset{0.496}{\longrightarrow} a_3 \overset{0.536}{\longrightarrow} a_1 \overset{0.488}{\longrightarrow} a_4.$$

The *max-product compositions* of the relations X and Y are:

$$Z^1 = X \circ Y = \begin{bmatrix} 0.665 & 0.475 & 0.380 & 0.475 \\ 0.63 & 0.45 & 0.36 & 0.45 \\ 0.525 & 0.375 & 0.3 & 0.375 \\ 0.35 & 0.25 & 0.2 & 0.25 \end{bmatrix}$$

$$Z^2 = Y \circ X = \begin{bmatrix} 0.25 & 0.275 & 0.35 & 0.475 \\ 0.35 & 0.385 & 0.49 & 0.45 \\ 0.4 & 0.44 & 0.56 & 0.76 \\ 0.35 & 0.385 & 0.49 & 0.665 \end{bmatrix}$$

The fuzzy relations Z^1, Z^2 are max-Δ transitive ones according to Proposition 1. The aggregated relation R computed with the help of (16), t-norm $T = \min(x, y)$ and t-conorm $S = \max(x, y)$ is:

$$R = \begin{bmatrix} 0.5 & 0.476 & 0.38 & 0.475 \\ 0.6 & 0.5 & 0.4 & 0.45 \\ 0.525 & 0.4 & 0.5 & 0.4 \\ 0.35 & 0.385 & 0.4 & 0.5 \end{bmatrix}$$

The antisymmetrized relation R' of R is:

$$R' = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.6 & 0.45 \\ 0 & 0.5 & 0.525 & 0.4 \\ 0 & 0 & 0.5 & 0.475 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \end{array}$$

The corresponding fuzzy linear ordering is:

$$a_2 \overset{0.4}{\longrightarrow} a_3 \overset{0.525}{\longrightarrow} a_1 \overset{0.475}{\longrightarrow} a_4.$$

The result orderings coincide in the considered cases.

Compare these results with the following ones obtained by a transformation of the relation W into some weighted coefficients $w^1, w^2, \ldots, w^m$ of the m-th criteria. As shown in (Chiclana et al. 2001), the vector of the weighted coefficients of a consistent fuzzy preference relation induces the same ordering among the set of criteria as the vector of quantifier guided dominance degrees, no matter which linguistic quantifier is used. For this reason, we propose to calculate the importance of the k-th criterion as the total sum of the values of the row k of the matrix W, i.e. $w^k = \sum_{j=1}^{m} w_{kj}$. The normalized vector for this matrix in our example is $W = (0.55,\ 0.45)$.

(a) Using the minimum operator to transform the preference values r_{ij}^k under the weighted coefficient w^k to generate a new value, the aggregated fuzzy preference relation Robtained by the maximum operator is:

$$R = \max(\min(0.55, X), \min(0.45, Y)) = \begin{bmatrix} 0.5 & 0.55 & 0.55 & 0.55 \\ 0.45 & 0.5 & 0.55 & 0.55 \\ 0.45 & 0.45 & 0.5 & 0.55 \\ 0.45 & 0.45 & 0.4 & 0.5 \end{bmatrix}$$

The antisymmetrized relation$R'$of R produces the following fuzzy linear ordering:

$$a_1 \xrightarrow{0.55} a_2 \xrightarrow{0.55} a_3 \xrightarrow{0.55} a_4.$$

(b) Using the t-norm T =min (x, y) to combine $r_{ij}^k$with w^k and t-conorm $S = \max$ (x, y) to aggregate the new computed relations, one has:

$$R = S(T(0.55, X), T(0.45, Y)) = \begin{bmatrix} 0.4381 & 0.3967 & 0.4404 & 0.5926 \\ 0.4845 & 0.4381 & 0.4731 & 0.6086 \\ 0.4656 & 0.4105 & 0.4381 & 0.5711 \\ 0.3338 & 0.2676 & 0.2927 & 0.4381 \end{bmatrix}.$$

The following fuzzy linear ordering from $R'$is obtained:

$$a_2 \xrightarrow{0.4731} a_3 \xrightarrow{0.4656} a_1 \xrightarrow{0.5926} a_4.$$

(c) Using the fuzzy linguistic quantifier "*most of*", the aggregated fuzzy preference relation obtained by corresponding OWA operator, is

$$R = \Phi_{most}^{W}(\langle 0, 55, X\rangle, \langle 0.45, Y\rangle) = \begin{bmatrix} 0.5 & 0.4 & 0.4 & 0.56 \\ 0.6 & 0.5 & 0.5 & 0.66 \\ 0.6 & 0.5 & 0.5 & 0.66 \\ 0.5 & 0.53 & 0.34 & 0.5 \end{bmatrix}.$$

The antisymmetrized relation$R'$of R produces the following fuzzy linear ordering:

$$a_2 \xrightarrow{0.5} a_3 \xrightarrow{0.6} a_1 \xrightarrow{0.56} a_4.$$

It is obvious that, the ordering for case (a) quite differs from the others due to the exceptional influence of the weighted coefficient on the aggregated relation. In case (b), the reduction of the membership degrees of R is a consequence of multiplication. The initial weighted coefficients are used only by the permutation in case (c) without taking into account their particular values. The ordering obtained by the suggested here approach is the same for cases (b) and (c) and is computed without transforming the relation W into some weighted coefficients, i.e. without reducing the initial information.

5 Concluding remarks

The composition of two fuzzy preference relations in an aggregation procedure is investigated. It is proved, that the composition is max-Δ transitive if both relations are additive or max–max transitive. It shows that using the composition to aggregate relations has practical application. A combination of t-norm and t-conorm is studied for obtaining a new fuzzy preference relation including the computed composition. This relation connects the relations on alternatives with the fuzzy preference relation between the criteria importance. It is proved that the new fuzzy preference relation preserves the max-Δ transitivity of the composition under some defined conditions, thus making possible to decide the problem of the alternatives ordering. The numerical example shows the result orderings by different compositions and different t-norms and t-conorms. This procedure for aggregation uses the whole information up to the final step of problem solution. The new relation possesses the transitivity property, which is important for the solution of the alternatives ranking problem. It can be easily performed on computer, when the number of criteria and alternatives is small.

Acknowledgments This work is supported by the Bulgarian Academy of Sciences under grant No 010077/2007.

References

Alonso, S., Chiclana, F., Herrera, F., Herrera-Viedma, E., Alcala-Fdez, J., & Porcel, C. (2005). A consistency based procedure to estimate missing pairwise preference values. *International Journal of Intelligent Systems*, http://decsai.ugr.es/~viedma/public.html.

Basile, L. (1990). Ranking alternatives by weak transitivity relation. In J. Kacprzyk & M. Fedrizzi (Eds.), *Multiperson decision making using fuzzy sets and possibility theory* (pp. 105–112). Netherlands: Kluwer.

Calvo, T., & Mesiar, R. (2003). Aggregation operators: Ordering and bounds. *Fuzzy Sets and Systems, 139*, 685–697.

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (1998). Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations. *Fuzzy Sets and Systems, 97*, 33–48.

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (2000). The ordered weighted geometric operator: Properties and applications. In: Proc. of 7th IPMU'2000, Int. Conf. on Inf. Proc. and Manag. of Univ. in Knowledge-Bases Systems, IPMU'2000, vol. II, DECSAI University of Granada, 2000, pp. 985–991.

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (2001). Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations. *Fuzzy Sets and Systems, 122*, 277–291.

Chiclana, F., Herrera, F., Herrera-Viedma, E., & Martinez, L. (2003). A note on the reciprocity in the aggregation of fuzzy preference relations using OWA operators. *Fuzzy Sets and Systems, 137*, 71–83.

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (2004). Rationality of induced ordered weighted operators based on the reliability of the source of information in group decision-making. *Kybernetika, 40*, 121–142.

Detyniecki, M. (2001). Mathematical aggregation operators and their application to video querying. Thesis for the degree Docteur de l'Universite Paris VI, www.lip6.fr/reports/lip6.2001.html.

Drewniak, J., & Dudziak, U. (2004). Safe transformations of fuzzy relations. In: Proceedings of EUROFUSE 2004, The 8-th Meeting of EURO Working Group on Fuzzy Sets, Workshop on Data and Knowledge Engineering, Sept. 22–25, 2004, Warszawa, Poland, pp. 195–203.

Dubois, D., & Prade, H. (1980). Fuzzy sets and systems: Theory and applications. New York: Academic Press.

Ekel, P. Y. (2002). Fuzzy sets and models of decision making. *International Journal of Computers and Mathematics with Applications, 44*(7), 863–875.

Fodor, J. C., & Roubens, M. (1995). Characterization of weighted maximum and some related operations. *Information Science, 84*, 173–180.

Grabisch, M., Orlowski, S., & Yager, R. (1998). Fuzzy aggregation of numerical preferences. In: R. Slowinski (Ed.), Fuzzy sets in decision analysis operations research and statistics, The handbooks of fuzzy sets series. (pp. 31–68) Boston: Kluwer.

Herrera-Viedma, E., Herrera, F., Chiclana, F., & Luque, M. (2004). Some issues on consistency of fuzzy preference relations. *European Journal of Operational Research, 154*, 98–109.

Jacas, J., & Recasens, J. (2003). Aggregation of t-transitive relations. *International Journal of Intelligent Systems, 18*(12), 1193–1214.

Jin, Y., & Sendhoff, B. (2002). Incorporation of fuzzy preferences into evolutionary multiobjective optimization. In: Proceedings of the 4-th Asia-Pacific Conference on Simulated Evolution and Learning, Singapore, Nov. 2002, pp. 26–30.

Kacprzyk, J. (1986). Group decision making with a fuzzy linguistic majority. *Fuzzy Sets and Systems, 18*, 105–118.

Ma, J., Fan, Z. P., & Huang, L. H. (1999). A subjective and objective integrated approach to determine attribute weights. *European Journal of Operational Research, 112*, 394–404.

Ma, J., Fan, Z. P., Jiang, Y. P., Mao, J. Y., & Ma, L. (2006). A method for repairing the inconsistency of fuzzy preference relations. *Fuzzy Sets and Systems, 157*, 2033.

Mesiar, R. (2003). Aggregation operators 2003. In: Proceedings of 3-rd Conference Of European Society for fuzzy Logic and Technology , *EUSFLAT 2003*, Sept. 10–12, Zittan, Germany, pp. 277–280.

Mesiar, R., & Saminger, S. (2004). Domination of ordered weighted averaging operators over t-norms. *Soft Computing, 8*, 562–570.

Nakamura, K. (1986). Preference relations on a set of fuzzy utilities as a basis for decision making. *Fuzzy Sets and Systems, 20*, 147–162.

Nguyen, H. T., & Walker, E. A. (2000). Fuzzy logic. London: Chapman & Hall/CRC Press.

Peneva, V., & Popchev, I. (2003). Properties of the aggregation operators related with fuzzy relations. *Fuzzy Sets and Systems, 139*(3), 615–633.

Peneva, V., & Popchev, I. (2005). Weighted transformation in aggregation operators uniting fuzzy relations. *Advanced Studies in Contemporary Mathematic, 10*(1), 25–44.

Roubens, M. (1989). Some properties of choice functions based on valued binary relations. *European Journal of Operation Research, 40*, 309–321.

Roubens, M. (1995). Fuzzy sets in preference modeling in decision analysis. In: Proceedings of VI IFSA World Congress, Sao Paulo, Brazil, 1995, pp. 19–23.

Sousa, J., & Kaymak, U. (2001). Model of predicative control using fuzzy decision function. *IEEE Transactions on SMC-Part B: Cybernetics, 31*(1), 54–65.

Tanino, T. (1984). Fuzzy preference orderings in group decision making. *Fuzzy Sets and Systems, 12*, 117–131.

Tanino, T. (1988). Fuzzy preference relations in group decision making. In: J. Kacprzyk, M. Roubens, (Eds.), Non-conventional preference relations in decision making. Berlin: Springer-Verlag.

Venugopalan, P. (1992). Fuzzy ordered Sets. *Fuzzy Sets and Systems, 46*, 221–226.

Xu, Z. S. (2004a). Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation. *International Journal of Approximate Reasoning, 36*(3), 261–270.

Xu, Z. S. (2004b). On compatibility of interval fuzzy preference matrices. *Fuzzy Optimization and Decision Making, 3*, 217–225.

Xu, Z. S. (2005). On method for uncertain multiple attribute decision making problems with uncertain multiplicative preference information on alternatives. *Fuzzy Optimization and Decision Making, 4*, 131–139.

Xu, Z., & Da, Q. (2003). An approach to improving consistency of fuzzy preference matrix. *Fuzzy Optimization and Decision Making, 2*, 3–12.

Yager, R. R. (1978). Fuzzy decision making including unequal objectives. *Fuzzy Sets and Systems, 1*, 87–95.

Yager, R. R. (1994). On weighted median aggregation. *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems, 2*(1), 101–114.

Yager, R. R. (1996). On mean type aggregation. *IEEE Transactions on SMC. Part B 26*(2), 209–221.

Zadeh, L. A. (1971). A similarity relations and fuzzy orderings. *Infomation Science, 3*, 177–200.

Zimmermann, H.-J. (1993). *Fuzzy set theory and its applications*. Kluwer Academic Publishers: Noewell, MA.