MATHEMATIQUES

Sous-ensemble flous

FUZZY CRITERIA IMPORTANCE WITH WEIGHTING FUNCTIONS

Vania Peneva, Ivan Popchev

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Abstract

The fuzzy criteria are given as fuzzy relations in multicriteria decision making problem. The weights of the criteria are parametrical linear functions (weighting functions) of the membership degrees of the preference relations between the alternatives. New transformed membership degrees are computed using the weighting functions. The properties of these new fuzzy relations required to decide the multicriteria decision making problems to be proved.

Key words: multicriteria decision making, fuzzy relations, aggregation operators, fuzzy relations' properties, weighting function

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1. Introduction. The models supporting the decision making by several criteria are based on the set of alternatives characterized with the help of a set of criteria together with their relative importance (weights). Developing procedures to determine the weights have become the aim for a lot of researches [1, 4, 8–10]. Weighting functions will be considered in this investigation instead of constant weights. These functions depend continuously on the criterion satisfaction values.

They have to possess some properties preserving the monotonicity and sensitivity of the alternative values by the criteria required from the aggregation procedures.

Consider the following multicriteria decision making problem. A finite set of alternatives is evaluated by several fuzzy criteria. The usual assumption is that criteria are independent. These criteria evaluate the alternatives comparing the couple of them, i.e. they assign fuzzy relations with membership degrees values in the unit interval. The importance of each criterion is a weighting function with values in unit interval as well. In order to compute the multicriteria score of the fuzzy relations an aggregation operator uniting the membership degrees by all relations (fuzzy criteria) taking into account the respective weighting functions will be used. The purpose is to aggregate the obtained weighted fuzzy relations in order to get a union relation as a fuzzy one, giving a possibility to decide a ranking, choice or cluster problem.

The concept of using weighting functions that depend continuously on the criterion satisfaction values (i.e. good or bad criteria performances) is suggested in [8, 9]. A model with weighting functions and its advisability is considered in [7]. A more complex type of such functions is used in this investigation.

Let the finite set of alternatives $A = \{a, b, c, \ldots, n\}$ be evaluated by fuzzy criteria $K = \{k_1, \ldots, k_m\}$, which compare the couples of alternatives and assign membership degrees to fuzzy relations corresponding to the criteria. Let also $\{R_1, \ldots, R_i, \ldots, R_m\}$ be the matrices of these fuzzy relations, i.e. $R_i = \|\mu_i(a, b)\|$, $i = 1, \ldots, m$, $\forall a, b \in A$, where $\mu_i(a, b)$ is the membership degree of the couple of alternative a, $b \in A$ to the fuzzy relation R_i (R_i means a fuzzy relation and a matrix corresponding to this relation as well, for simplicity). The weights of the criteria are given as functions $W = \{w_1(R_1), \ldots, w_m(R_m)\}$ of the membership degrees to each relation. An approach for uniting fuzzy relations is to use aggregation procedures which may be realized with the help of aggregation operators.

Taking into account the conception in [11], each of the membership degrees may be transformed using weighting functions of the criteria as follows:

(1) $g(w_i(\mu_i(a, b)), \mu_i(a, b)) = \tilde{\mu}_i(a, b), \ a, b \in A, \ i = 1, \ldots, m,$

and then the weighted aggregation is obtained as

(2) $Agg(\tilde{\mu}_1(a, b), \ldots, \tilde{\mu}_m(a, b)) = \mu^w(a, b),$

where Agg denotes an aggregation operator [2] and the function g satisfies the following properties [11]:

$x > y \rightarrow g(w, x) \geq g(w, y); \quad g(w, x)$ is monotone in $w; \ g(0, x) = id, \ g(1, x) = x,$

with the identity element, id, such that it does not change the aggregated value by adding it to the aggregation. The form of g depends on the type of aggregation being performed, e.g. it may be t-norm or t-conorm [5, 6, 11], taking into account their

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properties. The purpose of the following investigation is to prove the properties of the new relations with membership functions $\tilde{\mu}_i(a, b)$, $a, b \in A$, $i = 1, \ldots, m$ with the product t-norm for g and Weighted Mean operator for Agg.

2. Parametric linear weighting function. Let the membership degrees from the comparison of the alternatives $a, b \in A$ be $\mu_1(a, b) = x_1, \mu_2(a, b) = x_2, \ldots, \mu_m(a, b) = x_m$. The Weighted Mean operator in this case is defined as [8]

$$(3) \quad \mu^w(a, b) = \frac{\sum\limits_{i=1}^{m} f_i(x_i) x_i}{\sum\limits_{j=1}^{m} f_j(x_j)}, \quad \text{i.e. } \mu^w(a, b) = \sum\limits_{i=1}^{m} w_i(x_i) x_i \text{ with weighting function}$$

$$(4) \quad w_i(x_i) = \frac{f_i(x_i)}{\sum\limits_{j=1}^{m} f_j(x_j)} \quad \text{and the normalization condition } \sum\limits_{i=1}^{m} w_i(x_i) = 1.$$

The continuous functions $f_i(x)$, $i = 1, \ldots, m$ are defined in the unit interval for $x \in [0, 1]$ and they have continuous derivatives $f_i'(x)$, $i = 1, \ldots, m$ in this interval. It is proved [8] that the sufficient condition for the strict monotonicity of the operator (3) is $f_i'(x) \le f_i(x)$, $i = 1, \ldots, m$, $x \in [0, 1]$.

The functions $F_i(x) = \alpha_i f_i(x)/f_i(1)$ with $f_i(x) = 1 + \beta_i x$ and parameters $0 < \alpha_i \le 1$, $0 \le \beta_i \le 1$, $i = 1, \ldots, m$ will be used in this investigation [8]. Therefore,

$$(5) \qquad\qquad F_i(x) = \alpha_i \frac{1 + \beta_i x}{1 + \beta_i}.$$

It is obvious that $0 < F_i(0) = \alpha_i/(1 + \beta_i) \le F_i(1) = \alpha_i \le 1$, i.e. parameter α_i controls the value $F_i(1) = \alpha_i$, when the criteria satisfaction value is one. The parametric choice β_i controls the ratio between the largest and smallest values of the function (5), when the criteria satisfaction values are zero and one, i.e. $1 \le F_i(1)/F_i(0) = 1 + \beta_i \le 2$, $i = 1, \ldots, m$. Taking into account (3), (4) and (5) one has

$$(6) \qquad\qquad \mu^w(a, b) = \frac{\sum\limits_{i=1}^{m} F_i(x_i) x_i}{\sum\limits_{j=1}^{m} F_j(x_j)} = \frac{\sum\limits_{i=1}^{m} \gamma_i(1 + \beta_i x_i) x_i}{S(a, b)}, \quad \text{where}$$

$$(7) \quad S(a, b) = \sum\limits_{i=1}^{m} F_i(x_i) = \sum\limits_{i=1}^{m} \alpha_i \frac{1 + \beta_i x_i}{1 + \beta_i} = \sum\limits_{i=1}^{m} \gamma_i(1 + \beta_i x_i), \quad \text{with } \gamma_i = \frac{\alpha_i}{1 + \beta_i},$$

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$$(8) \qquad w_i(x_i) = \frac{F_i(x_i)}{\sum\limits_{j=1}^{m} F_j(x_j)} = \frac{\gamma_i(1 + \beta_i x_i)}{S(a,b)}.$$

3. Some properties of the weighted relations. Taking into account the Weighted Mean operator (6) and the product t-norm for g in (1), the new membership degrees of the weighted relations are

$$(9) \quad \tilde{\mu}_i(a,b) = \begin{cases} 1 & \text{if } a = b \\ w_i(\mu_i(a,b)) \times \mu_i(a,b) & \text{if } a \neq b \end{cases} \qquad \forall a, b \in A, \; i = 1, \ldots, m.$$

The properties of these relations connected with the properties of reflexivity, symmetry and transitivity will be studied. The membership functions (9) show that these weighted relations preserve the reflexive and symmetrical properties of the initial relations. But the max-min transitivity property is not preserved, as it is seen from examples.

Proposition 3.1. If the relations $R_i = \|\mu_i(a,b)\|$, $i = 1, \ldots, m$, $\forall a, b \in A$ are max-min transitive, then (9) transforms them to max-Δ transitive fuzzy relations.

Proof. Let R_i be max-min transitive relations, i.e.
$\mu_i(a,c) \geq \min(\mu_i(a,b), \mu_i(b,c))$, $\forall a, b, c \in A$, $i = 1, \ldots, m$. Introducing the notations $\mu_i(a,c) = z_i$, $\mu_i(a,b) = x_i$, $\mu_i(b,c) = y_i$, the above inequality becomes

$$(10) \qquad z_i \geq \min(x_i, y_i), \quad i = 1, \ldots, m.$$

It has to be proved that (see (9))

$$(11) \qquad \tilde{\mu}_i(a,c) \geq \max(0, \tilde{\mu}_i(a,b) + \tilde{\mu}_i(b,c) - 1), \; \forall a, b, c \in A, \; i = 1, \ldots, m,$$

or more precisely

$$(12) \qquad \frac{\gamma_i(1 + \beta_i z_i) z_i}{S(a,c)} \geq \max\left(0, \frac{\gamma_i(1 + \beta_i x_i) x_i}{S(a,b)} + \frac{\gamma_i(1 + \beta_i y_i) y_i}{S(b,c)} - 1\right),$$

where according to (7)

$$S(a,b) = \sum_{i=1}^{m} \gamma_i(1 + \beta_i x_i); \quad S(b,c) = \sum_{i=1}^{m} \gamma_i(1 + \beta_i y_i); \quad S(a,c) = \sum_{i=1}^{m} \gamma_i(1 + \beta_i z_i).$$

— If $x_i + y_i \leq 1$, then $w_i(x_i)x_i + w_i(y_i)y_i \leq 1$, i.e. (12) holds.
— If $x_i + y_i > 1$, then it has to be proved that

$$(13) \qquad \frac{\gamma_i(1 + \beta_i z_i) z_i}{S(a,c)} \geq \frac{\gamma_i(1 + \beta_i x_i) x_i}{S(a,b)} + \frac{\gamma_i(1 + \beta_i y_i) y_i}{S(b,c)} - 1.$$

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According to (10), z_i can be preordered in such a way that

$$x_i \leq z_i \leq y_i \quad \text{for} \quad i = 1, \ldots, k, \quad y_i \leq z_i \leq x_i \quad \text{for} \quad i = k+1, \ldots, m.$$

Let $j \in [1, k]$, i.e. $x_j \leq z_j \leq y_j$. Introducing the notations $(1 + \beta_j z_j)z_j = Z_j$, $(1 + \beta_j x_j)x_j = X_j$, $(1 + \beta_j y_j)y_j = Y_j$, (13) becomes

(14)
$$\frac{\gamma_j Z_j}{S(a,c)} \geq \frac{\gamma_j X_j}{S(a,b)} + \frac{\gamma_j Y_j}{S(b,c)} - 1.$$

If
$$\frac{Z_j}{S(a,c)} \geq \frac{X_j}{S(a,b)}$$

or
$$\frac{Z_j}{S(a,c)} \geq \frac{Y_j}{S(b,c)},$$

then
$$\frac{\gamma_j Z_j}{S(a,c)} \geq \frac{X_j}{S(a,b)} \frac{Y_j}{S(b,c)} \geq \max\left(0, \frac{\gamma_j X_j}{S(a,b)} + \frac{\gamma_j Y_j}{S(b,c)} - 1\right)$$

and (12) is proved.

The more complicated case is when $\dfrac{\gamma_j Z_j}{S(a,c)} < \dfrac{X_j}{S(a,b)} \dfrac{Y_j}{S(b,c)}$, i.e.

(15)
$$S(a,b)Z_j < S(a,c)X_j \quad \text{and} \quad S(b,c)Z_j < S(a,c)Y_j.$$

From $j \in [1, k]$, i.e. $x_j \leq z_j \leq y_j$ follows that $X_j \leq Z_j$ and therefore $S(a,c) > S(b,c) > S(a,b)$, for example. According to (14) one has to prove that

$$\frac{\gamma_j Z_j}{S(a,c)} + 1 \geq \frac{\gamma_j X_j}{S(a,b)} + \frac{\gamma_j Y_j}{S(b,c)}, \quad \text{i.e.}$$

(16) $\quad S(a,b)S(b,c)[\gamma_j Z_j + S(a,c)] \geq S(b,c)S(a,c)\gamma_j X_j + S(a,b)S(a,c)\gamma_j Y_j.$

But from (15) one has

(17) $S(a,b)S(b,c)S(a,c) - S(a,b)S(a,c)\gamma_j Y_j = S(a,b)S(a,c)[S(b,c) - \gamma_j Y_j] \geq 0$

and

(18) $S(a,b)S(b,c)\gamma_j Z_j - S(b,c)S(a,c)\gamma_j X_j = \gamma_j S(b,c)[S(a,b)Z_j - S(a,c)X_j] \leq 0.$

Compare (17) and (18). If (17) is greater or equal to (18), then (13) is proved. The results from the comparisons of the separated multipliers are $S(a,c) > S(b,c)$, $S(a,b) > \gamma_j$. According to the third multipliers one has

(19) $\quad S(b,c) - \gamma_j Y_j = \displaystyle\sum_{i=1}^{j-1} \gamma_i(1 + \beta_i y_i) + \gamma_j(1 + \beta_j y_j)(1 - y_j) + \sum_{i=j+1}^{m} \gamma_i(1 + \beta_i y_i)$

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$$S(a,c)X_j - S(a,b)Z_j = \sum_{i=1}^{j-1} [\gamma_i(1+\beta_i z_i)(1+\beta_j x_j)x_j] + \gamma_j(1+\beta_j z_j)(1+\beta_j x_j)x_j$$

$$(20) \quad + \sum_{i=j+1}^{m} [\gamma_i(1+\beta_i z_i)(1+\beta_j x_j)x_j] - \sum_{i=1}^{j-1} [\gamma_i(1+\beta_i x_i)(1+\beta_j z_j)z_j]$$

$$-\gamma_j(1+\beta_j x_j)(1+\beta_j z_j)z_j - \sum_{i=j+1}^{m} [\gamma_i(1+\beta_i x_i)(1+\beta_j z_j)z_j].$$

Compare the addends from (19) and (20), separately. One has

$$\sum_{i=1}^{j-1} \gamma_i(1+\beta_i y_i) \geq \sum_{i=1}^{j-1} \gamma_i(1+\beta_i z_i)(1+\beta_j x_j)x_j - \sum_{i=1}^{j-1} \gamma_i(1+\beta_i x_i)(1+\beta_j z_j)z_j,$$

because

$$(21) \qquad \gamma_i(1+\beta_i y_i) \geq \gamma_i(1+\beta_i z_i)(1+\beta_j x_j)x_j - \gamma_i(1+\beta_i x_i)(1+\beta_j z_j)z_j$$

which follows from inequality (21) equivalent to

$$1+\beta_i y_i + z_j + \beta_i x_i z_j + \beta_j z_j^2 + \beta_i \beta_j x_i z_j^2 \geq x_j + \beta_i z_i x_j + \beta_j x_j^2 + \beta_i \beta_j z_i x_j^2,$$

and $x_i \leq z_i \leq y_i$ for $i = 1, \ldots, k$, $j \in [1,k]$, $\beta_i y_i \geq \beta_i z_i x_j$, $\beta_j z_j^2 \geq \beta_j x_j^2$,

$$1+\beta_i x_i z_j + \beta_i \beta_j x_i z_j^2 \geq \beta_i \beta_j z_i x_j^2.$$

Besides,

$$\gamma_j(1+\beta_j y_j)(1-y_j) \geq \gamma_j(1+\beta_j z_j)(1+\beta_j x_j)x_j - \gamma_j(1+\beta_j x_j)(1+\beta_j z_j)z_j,$$

because $(1+\beta_j y_j)(1-y_j) \geq (1+\beta_j z_j)(1+\beta_j x_j)(x_j - z_j)$, taking into account that the left side of the above inequality is positive, but the right side is negative $(x_j \leq z_j)$.
    Finally,

$$\sum_{i=j+1}^{m} \gamma_i(1+\beta_i y_i) \geq \sum_{i=j+1}^{m} \gamma_i(1+\beta_i z_i)(1+\beta_j x_j)x_j - \sum_{i=j+1}^{m} \gamma_i(1+\beta_i x_i)(1+\beta_j z_j)z_j,$$

because: if $i \leq k$, the case is analogical to the case, when $i \in [1, j-1]$; if $i > k$, then $y_i \leq z_i \leq x_i$.

Therefore $x_j \leq z_j$, $\beta_i x_i z_j \geq \beta_i z_i x_j$, $\beta_j z_j^2 \geq \beta_j x_j^2$, $1 + \beta_i y_i + \beta_i \beta_j x_i z_j^2 \geq \beta_i \beta_j z_i x_j^2$, i.e. the inequality (13) holds and hence inequality (12) is proved.

Proposition 3.2. If R_i is a reflexive and max-min transitive relation, then R_i^w with membership function (9) is a fuzzy preorder.

The proof follows from Proposition 3.1 and definition of a fuzzy preorder in [3].

Proposition 3.3. If R_i is a similarity relation (reflexive, symmetrical and max-min transitive relation), then R_i^w with membership function (9) is a likeness (reflexive, symmetrical and max-Δ transitive) relation.

The proof follows from Proposition 3.1 and definitions given in [3].

Proposition 3.4. [5] If R_i^w, $i = 1, \ldots, m$ are fuzzy preorder relations, then aggregation relation with a membership function (3) is a fuzzy preorder relation as well.

Proposition 3.5. [5] If R_i^w, $i = 1, \ldots, m$ are likeness relations, then aggregation relation with a membership function (3) is a likeness relation as well.

4. Conclusion. Using weighting functions instead of constant weights of the fuzzy criteria that assign fuzzy relations between the alternatives produces weighted aggregation with complex dependency on the membership degrees. The advantage of these functions is in their ability to fine the small values and to reward the great values of the membership degrees. The proved properties of the weighted relations give a possibility to use transformed relations in aggregation procedures. The main contribution of the aggregation preorder relation consists in the decision making problems for choice or ordering among the set of alternatives. The property of likeness of the aggregated relation in this case is useful for solving clustering problem of the alternatives set.

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Institute of Information Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2
1113 Sofia, Bulgaria

V. Peneva, I. Popchev