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AGGREGATION OF FUZZY RELATIONS WITH FUZZY WEIGHTED COEFFICIENTS[1]

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Abstract. Aggregation of fuzzy relations in multicriteria decision making problems is considered. Pair wise comparisons between the alternatives by several criteria are given as fuzzy relations with definite properties. The weighted coefficients of the criteria are fuzzy numbers. The aim is to suggest models for aggregation of the fuzzy relations taking into account the criteria importance. The properties of the aggregated relation need to guarantee the decision making problems solution. Illustrative examples are given for comparison of the suggested models.

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1. Introduction

Fuzzy relations and fuzzy numbers are the basic concept in the following multicriteria decision making problem. Let $A=\{a_1,\dots,a_n\}$ be the finite set of alternatives evaluated by several fuzzy criteria $K=\{k_1,\dots,k_m\}$, i.e. these criteria give fuzzy relations $\{R_1,R_2,\dots,R_m\}$ between each couple of alternatives. When the cardinality n of A is small, the relations may be represented by the $n\times n$ matrices $R_k=\|\mu_k(a_i,a_j)\|,\quad i,j=1,\dots,n,\ k=1,\dots,m$, where $\mu_k: A\times A\to[0,1]$ is the membership function of the relation R_k and $\mu_k(a_i,a_j)$ is the membership degree to R_k obtained from comparing alternatives a_i and a_j by criterion k_k. If $\mu_k(a_i,a_j)=0.5$, this indicates indifference between a_i and a_j, $\mu_k(a_i,a_j)=1$ indicates that a_i is absolutely preferred to a_j, and $\mu_k(a_i,a_j)>0.5$ indicates that a_i is preferred to a_j by the k-th criterion. In this case, the preference matrices $R_k, k=1,\dots,m$ are usually assumed to be additively reciprocal, i.e.

$$\mu_k(a_i,a_j)+\mu_k(a_j,a_i)=1,\ i,j=1,\dots,n. \quad (1)$$

Let the set of weighted coefficients (weights) of the criteria $K=\{k_1,\dots,k_m\}$ consists of trapezoidal fuzzy numbers (t.f.n.) $\widetilde{W}=\{\widetilde{w}_1,\dots,\widetilde{w}_m\}$, i.e. $\widetilde{w}_k=(w_k^1,w_k^2,w_k^3,w_k^4), k=1,2,\dots,m$ and $w_k^1\le w_k^2\le w_k^3\le w_k^4$. The membership function of $\widetilde{w}_k$ is [14, 15]:

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$$\mu_{\tilde{w}_k}(x)=\begin{cases}0 & -\infty < x \le w_k^1\\ f(x) \quad w_k^1 \le x \le w_k^2, & f(x) \text{ is an increasing function}\\ 1 & w_k^2 \le x \le w_k^3\\ g(x) \quad w_k^3 \le x \le w_k^4, & g(x) \text{ is an decreasing function}\\ 0 & w_k^4 \le x < +\infty\end{cases} \tag{2}$$

The setting problem in this multicriteria decision making model is connected with aggregation of fuzzy relations by the individual criteria taking into account the given importance of the criteria. The aggregated relation has to possess properties required to solve problems for a choice of set of the "best" alternatives or to rank them from the "best" to the "worst" ones. The following models for solving this problem will be considered:

- the fuzzy numbers corresponding to the criteria importance are directly used in the computation of the aggregated relation,
- these numbers are transformed in an appropriated form for the aggregation.

The aggregation procedures have to be chosen according to the method for using fuzzy numbers. Aggregation operators may perform these procedures. Very good studies of the aggregation operators, by presenting the characteristics, the advantages and disadvantages of each operator and the relations between them, are available in [10, 19]. The choice of an operator for specific application depends on various factors [17], e.g. type of value scale used for given criterion: bipolar or not, quantitative or qualitative. In each case, some aggregation operators look more plausible or feasible than others. Some choice has to be made according to, e.g.:

- the mathematical model of the operators,
- the properties of the operators for deciding problems of ranking or choice,
- the sensitivity of the operators for small variations of their arguments.

2. Aggregation of sequences from fuzzy numbers

Let Agg denotes an aggregation operator. Each membership degree $\mu_k(a,b)$ to the relation R_k may be transformed using weights (fuzzy numbers) as follows [32]:

$$t(\tilde{\tilde{w}}_k, \mu_k(a,b)) = \tilde{\mu}_k(a,b),\ a,b \in A,\ i=1,\dots,m, \tag{3}$$

where t is or a t-norm and $\tilde{\tilde{w}}_k = \tilde{w}_k$ or t is a t-conorm and $\tilde{\tilde{w}}_k = 1-\tilde{w}_k$. Taking into account the operations with fuzzy numbers and different t-norms, it follows that $\tilde{\mu}_k(a,b)$ is a fuzzy number, as well. The aggregation of $\tilde{\mu}_k(a,b),\ k=1,\dots,m$ may be done with the help of aggregation operators [29]

$$Agg(\tilde{\mu}_1(a,b),\dots,\tilde{\mu}_m(a,b)) = \mu^w(a,b). \tag{4}$$

The methods for computing the aggregated fuzzy numbers are different depending on the selected operator and the type of fuzzy numbers [29].

Example 2.1. Consider two normalized fuzzy numbers, i.e. $w_k^s \in [0,1],\ k=1,\dots,m,\ s=1,\dots,4$,

$$\tilde{w}_1 = (0.4, 0.6, 0.8, 0.9) \text{ and } \tilde{w}_2 = (0.2, 0.3, 0.4, 0.7) \tag{5}$$

as weights of the fuzzy criteria k_1 and k_2, evaluating four alternatives by fuzzy relations R_1 and R_2. One can verify that these relations are additively transitive and reciprocal.

$$R_1 = \begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & 0.5 & 0.55 & 0.7 & 0.95 \\ a_2 & 0.45 & 0.5 & 0.65 & 0.9 \\ a_3 & 0.3 & 0.35 & 0.5 & 0.75 \\ a_4 & 0.05 & 0.1 & 0.25 & 0.5 \end{array} \quad R_2 = \begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & 0.5 & 0.3 & 0.2 & 0.3 \\ a_2 & 0.7 & 0.5 & 0.4 & 0.5 \\ a_3 & 0.8 & 0.6 & 0.5 & 0.6 \\ a_4 & 0.7 & 0.5 & 0.4 & 0.5 \end{array}, \qquad (6)$$

Using t-norm $t(x,y) = xy$ in (3) and the aggregation operator Weighted Mean

$$\mu^w(a_i,a_j) = \sum_{k=1}^{m} \tilde{w}_k \mu_k(a_i,a_j) \qquad (7)$$

one has: $\mu^w(a_1,a_2) = \tilde{w}_1\mu_1(a_1,a_2) + \tilde{w}_2\mu_2(a_1,a_2) = (0.28, 0.42, 0.56, 0.705)$.

3. Transformation of fuzzy numbers

There exist different methods for transformation of fuzzy numbers. These may be classified as:

- using ranking function,
- computing fuzzy relation between them.

3.1. Ranking function approach

The first approach [2] consists in a determination of a ranking function F mapping each fuzzy number onto the real line. This function is such that if $F(\tilde{w}_i) <,=,> F(\tilde{w}_j)$, then $\tilde{w}_i <,=,> \tilde{w}_j$. This approach has been followed by many authors, e.g. [2, 3, 11, 16, 22, 28, 39, etc]. The new weighted coefficients using these approaches are real numbers $F(\tilde{w}_i)$, $i = 1,...,m$. Some of these methods use an index of optimism α (degree of risk of the decision maker), i.e.

$$F(\tilde{w}_i) = \alpha F_1(\tilde{w}_i) + (1-\alpha) F_2(\tilde{w}_i), \alpha \in [0,1]. \qquad (8)$$

This index of optimism is useful in cases of weighted aggregation taking into account the basic condition $\sum_{i=1}^{m} F(\tilde{w}_i) = 1$, i.e. $\sum_{i=1}^{m} (\alpha F_1(\tilde{w}_i) + (1-\alpha)F_2(\tilde{w}_i)) = 1$. The coefficient α is computed as follows, then:

$$\alpha = \frac{1 - \sum_{i=1}^{m} F_2(\tilde{w}_i)}{\sum_{i=1}^{m} F_1(\tilde{w}_i) - \sum_{i=1}^{m} F_2(\tilde{w}_i)} \qquad (9)$$

If the condition for weighted aggregation is $\max_i\{F(\tilde{w}_i)\} = 1$ [32], then new weights have to be computed. For example, let $\max_i\{F(\tilde{w}_i)\} = F(\tilde{w}_k)$, then the newly computed weighted coefficients are: $F'(\tilde{w}_i) = \frac{F(\tilde{w}_i)}{F(\tilde{w}_k)}$, $i = 1,...,m$.

Example 3.1. Consider the two fuzzy numbers (5). One of the approaches with index of optimism is the G-index (geometrical index) [28]. According to it

$$F_1(\tilde{w}_i) = w_i^1 + \frac{(w_i^4 - w_i^1) + (w_i^3 - w_i^2)}{2} \times \frac{1}{[(w_i^4 - w_i^3)^2 + 1]^{1/2}}$$

$$F_2(\tilde{w}_i) = w_i^4 - \frac{(w_i^4 - w_i^1) + (w_i^3 - w_i^2)}{2} \times \frac{1}{[(w_i^2 - w_i^1)^2 + 1]^{1/2}}. \qquad (10)$$

Computed G-indexes for $\tilde{w}_1$ and $\tilde{w}_2$ are:
$F(\tilde{w}_1) = \alpha\ 0.748 + (1-\alpha)0.557$ and $F(\tilde{w}_2) = \alpha 0.4874 + (1-\alpha)0.402$. Then from (9) $\alpha = 0.1483$, $F(\tilde{w}_1) = 0.585$, $F(\tilde{w}_2) = 0.415$ and as it is seen $F(\tilde{w}_1) + F(\tilde{w}_2) = 1$.

If the condition $\max_i \{F(\tilde{w}_i)\} = 1$ is required, then $F'(\tilde{w}_1) = 1$, $F'(\tilde{w}_2) = 0.7094$.

3.2. Comparison index approach

A fuzzy relation between given fuzzy numbers is obtained by using these methods. They are based on the idea, that the property "a fuzzy number is not worst then another" is a linguistic property and every decision maker or expert measures such a property in a personal way. Methods for comparing couples of fuzzy numbers by computing the corresponding degrees of preference may be found in [4, 9, 12, 23, 24, 26, 40, etc.]

Let $W = \|w(\tilde{w}_i, \tilde{w}_j)\| = \|w_{ij}\|, i, j = 1,...,m$ be a fuzzy preference relation, where $w(\tilde{w}_i, \tilde{w}_j)$ is the preference degree between the fuzzy numbers $\tilde{w}_i$ and $\tilde{w}_j$ corresponding to the importance of criteria k_i and k_j. The relation W has to possess some properties required in the aggregation procedure. Some of these properties are e.g. reflexivity, reciprocity, transitivity. For example, the relation obtained in [9] is antireflexive, antisymmetrical and max-min transitive, the one computed by the approach suggested in [4] is reflexive and reciprocal.

Example 3.2. Compute the preference relation between the above fuzzy numbers (5) with the help of method suggested by Chen [4], because it gives a relation possessing properties required for the aggregation procedure, presented in section 4.2. Computations are made following the algorithm given in this paper:

1. Compute $\tilde{w}_1 - \tilde{w}_2 = (0.2, 0.3, 0.4, 0.2) = (b^1, b^2, b^3, b^4)$;
2. Let $\alpha_0 = 0, \alpha_1 = 0.25, \alpha_2 = 0.5, \alpha_3 = 0.75, \alpha_4 = 1$;
3. Compute the α-cuts of $\tilde{w}_1 - \tilde{w}_2$, i.e.

$$l_i = a_1 + \alpha_i(a_2 - a_1),\ \ r_i = a_4 + \alpha_i(a_3 - a_4),\ i = 1,2,3,4;$$

4. As it is seen, all $l_i > 0, r_i > 0, i = 1,...,4$ and therefore the negative sub-area of $\tilde{w}_1 - \tilde{w}_2$ is equal to 0. Hence, the preference degree of $\tilde{w}_1$ over $\tilde{w}_2$ is equal to 1. It is proved that this relation is reciprocal (1), that's why $W = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}$.

4. Weighted aggregation of fuzzy relations

The aggregation operators uniting fuzzy relations and the transformed weights of the criteria importance are used below. Methods using aggregation operators are different according to the transformed values of the fuzzy numbers.

4.1. Aggregation with crisp weighted coefficients

The aggregation operators may be separated in two groups in this case:
- the weighted coefficients present in their mathematical models;
- the weights do not present in the aggregation operators' formula.

The dependence between the properties of the aggregated relation R obtained with the help of the operators from the first group and the individual relations R_k, $k = 1,...,m$ is represented in [5, 6, 7, 13, 30].

The properties of the weighted transformed aggregation operators from the second group are investigated in [32]. Transformation of the membership degrees of the fuzzy relations taking into account the weights are made with the help of t-norms and corresponding t-conorms.

A generalization of t-norms and t-conorms is the parameterized t-norms and t-conorms. Some of the common t-norms and t-conorms can be obtained by varying the parameters in these parameterized norms. Three families of parameterized t-norms and t-conorms: the Hamacher, the Yager and the Weber-Sugeno are studied. The relationships between these norms and the common ones and their properties are given in [31].

Example 4.1. As it is computed in Example 3.1, the transformed fuzzy numbers (5) to the real ones are $F(\tilde{w}_1) = 0.585$, $F(\tilde{w}_2) = 0.415$. Using the operator Weighted Mean (7) and relations from (6), the aggregated degrees are, e.g.

$\mu^w(a_1,a_2) = F(\tilde{w}_1)\mu_1(a_1,a_2) + F(\tilde{w}_2)\mu_2(a_1,a_2) = 0.4463$ and so on.

4.2. Aggregation by a fuzzy preference relation between the criteria importance

Let W be the preference relation between the criteria importance and $\{R_1,R_2,...,R_m\}$ be the relations between the couples of alternatives. The aim is to use the whole information given above to the final stage of the problem solution, without transforming the relation W into some weighted coefficients.

To make a consistent choice or to rank the alternatives from the "best" to the "worst" one, when assuming fuzzy preference relations, a set of properties to be satisfied has been suggested. The consistency in this case has a direct effect on the ranking results of the final decision. The study of consistency is associated with the concept of transitivity [20]. Let $\mu : A \times A \to [0,1]$ be a membership function of a fuzzy relation and $a,b,c \in A$. Some transitivity properties are:

- Max-min transitivity [15]: $\mu(a,c) \geq \min(\mu(a,b),\mu(b,c))$;
- Max-max transitivity [36]: $\mu(a,c) \geq \max(\mu(a,b),\mu(b,c))$;
- Restricted max-min transitivity or moderate transitivity [36]:
  $\mu(a,b) \geq 0.5,\ \mu(b,c) \geq 0.5 \Rightarrow \mu(a,c) \geq \min(\mu(a,b),\mu(b,c))$;
- Restricted max-max transitivity [36]:
  $\mu(a,b) \geq 0.5,\ \mu(b,c) \geq 0.5 \Rightarrow \mu(a,c) \geq \max(\mu(a,b),\mu(b,c))$;
- Additive transitivity [36]: $\mu(a,c) = \mu(a,b) + \mu(b,c) - 0.5$.

Characterizations and comparisons between these transitivity properties are suggested in [20]. The additive transitivity is a stronger property than restricted max-max one, which is a stronger concept than the restricted max-min transitivity, but a weaker property than max-max transitivity. The latter property is a stronger one than max-min transitivity, which is a stronger property than restricted max-min transitivity. Methods for constructing fuzzy preference relations from preference data are described in [1, 18, 20, 38]. Applying these methods it is possible to get consistency of the fuzzy preference relations and thus avoid from inconsistent solutions in the decision making processes.

The dependences between the properties of the aggregated fuzzy preference relation and the ones of the individual relations for this case are investigated in [6, 7, 13, 21, 25, 30, 32, 35].

4.2.1. Method with new fuzzy preference relation

Let $\mu_i(a,b)$ be the membership degree from the comparison of the alternatives $a,b \in A$ to the fuzzy preference relation R_i. Taking into account the relation W (see section 3.2.), a new fuzzy relation $R_{ij}, i,j=1,\dots,m$ between R_i and $R_j, R_i \neq R_j$ with the following membership degrees is suggested:

$$r_{ij}(a,b)=\begin{cases} 0.5 & \text{if } a=b \\ S(T(w_{ij},\mu_i(a,b)),T(w_{ji},\mu_j(a,b))) & \text{if } a\neq b \end{cases} \quad (11)$$

where T is a t-norm and S is a corresponding t-conorm.

The core idea used in (11) is that the comparison operator "pessimistically" combines the two relations to obtain measures of match which can be after that "optimistically" united in an overall result. Thus, if a t-norm provides the "pessimistic" combination, a t-conorm is a suitable generalization of the concept of "optimistic" union. Since $R_{ij}=R_{ji}$, the number k of the new relations will be equal to the combinations of two elements over m, i.e. $k=\frac{m(m-1)}{1.2}$. Aggregation operators uniting these k relations can be used after that to obtain the aggregated fuzzy relation giving a possibility to decide the choice or ranking problems. The transitivity is one of the most important properties concerning preferences. The examples show that the relation (11) does not preserve the max-min transitivity. But this kind of transitivity is a very strong property imposed on a fuzzy relation according to [41]. Several useful definitions for transitivity are suggested, which are compared in [37]. The weakest of all these definitions is the max-Δ transitivity, i.e. $\mu(a,c)\geq \max(0,\mu(a,b)+\mu(b,c)-1), a,b,c\in A$. It is shown that this is the most suitable notion of transitivity for fuzzy ordering.

Proposition 4.2.1 [33] The relation (11) is max-Δ transitive if the relations $R_i, i=1,\dots,m$ are max-min transitive and the relation W is additive reciprocal.

4.2.2. Aggregation of fuzzy preference relations by composition

The composition of two relations in an aggregation procedure is used. If the composition possesses some properties required for solving the problems of ranking or choice, then it may be used in such procedures. This gives one practical application of the composition.

Definition [27]. Let X and Y be fuzzy relations in $A\times A$ and let T be a t-norm. The composition $X\circ Y$ of these relations with respect to T is the fuzzy relation on $A\times A$ with membership function

$$\mu(a_i,a_j)=\mu_{X\circ Y}(a_i,a_j)=\max_k\{T(\mu_X(a_i,a_k),\mu_Y(a_k,a_j))\},\ i,j,k=1,\dots,n. \quad (12)$$

When $T=\min$, then the composition is a max-min one. When $T=xy$, then it is a max-product composition. $X\circ Y$ can be obtained as the matrix product of X and Y. It has to be taken into account that $X\circ Y\neq Y\circ X$ in general.

Let $X = \|x_{ij}\|$ and $Y = \|y_{ij}\|$, $i, j = 1,...,n$ are fuzzy relations in $A \times A$, where x_{ij}, y_{ij} denote the membership degrees of the comparison of the alternatives $a_i, a_j \in A$ to the fuzzy preference relations X and Y, respectively. What kind of transitivity must possess both relations to have their composition some transitivity properties? The examples show that the composition does not preserve the transitivity properties, but it transforms the additive and max-max transitivity into the max-Δ one (see propositions 4.2.2, 4.2.3 below). Besides the composition of two restricted max-min or max-max transitivity relations is not always a max-Δ transitive relation.

Proposition 4.2.2. [34]. If two fuzzy preference relations are additively or max-max transitive, then the composition of these relations is max-Δ transitive.

Taking into account the relation *W*, a new fuzzy relation between *X* and *Y* with the following membership degrees is suggested:

$$r(a_i, a_j) = \begin{cases} 0.5 & \text{if } a_i = a_j \\ S(T(w^1, z_{ij}^1), T(w^2, z_{ij}^2)) & \text{if } a_i \neq a_j \end{cases}, \qquad (13)$$

where from (12) $Z^1 = X \circ Y = \|z_{ij}^1\|$, $z_{ij}^1 = \max_k \{T(x_{ik}, y_{kj})\}$,

$Z^2 = Y \circ X = \|z_{ij}^2\|$, $z_{ij}^2 = \max_k \{T(y_{ik}, x_{kj})\}$, $k = 1,2,...,n$,

$w^1 = w(k_X, k_Y)$, $w^2 = w(k_Y, k_X)$ are the preference degrees to the criterion with a relation *X* over *Y* and *Y* over *X*, respectively, *T* is a t-norm and *S* is a corresponding t-conorm.

The main idea used in (13) is that the composition of two relations compares the preference degrees of the i-th alternative to all alternatives from the first relation with the preference degrees of the all alternatives to the j-th alternative from the second relation and vice versa, because the operation composition is not commutative. Then taking into account the relation *W*, i.e. that w^1 and w^2 are the preference degrees of the relation *X* over *Y* and *Y* over *X*, respectively, a comparison operator is used that "pessimistically" combines the relations *W* and Z^1, *W* and Z^2 to obtain measures of match which can be after that "optimistically" united in an overall result. Aggregation operators uniting these $k = \frac{m(m-1)}{1.2}$ relations can be used after that to obtain the aggregated fuzzy relation. The aim in this method is to use the whole information given above up to the final stage of the problem solution without transforming the relation W into some weighted coefficients. The following proposition is essential for this purpose.

Proposition 4.2.3 [34]. If the relations Z^1, Z^2 are max-Δ transitive ones and the relation W is additive reciprocal, then the relation (13) is max-Δ transitive for the couple of t-norms ($T = \min$, $S = \max$) and ($T = xy$, $S = x + y - xy$).

5. Aggregated relations and orderings from different methods

The aggregated relations *R* are computed for weighted coefficients (5), relations (6) and aggregation operator Weighted Mean (7) using data from the above examples.

5.1. Aggregation of fuzzy numbers

The fuzzy preference relation **R** (the matrix from fuzzy numbers) is obtained according to the method supposed in section 2 (see Example 2.1):

$$R=\begin{bmatrix} 0.5 & (0.28,0.42,0.56,0.705) & (0.3,0.45,0.64,0.77) & (0.44,0.66,0.88,1.065) \\ (0.32,0.48,0.64,0.895) & 0.5 & (0.34,0.51,0.68,0.86) & (0.46,0.69,0.92,1.16) \\ (0.28,0.42,0.56,0.83) & (0.28,0.39,0.52,0.74) & 0.5 & (0.42,0.63,0.84,1.095) \\ (0.16,0.24,0.32,0.535) & (0.14,0.21,0.28,0.44) & (0.18,0.33,0.36,0.505) & 0.5 \end{bmatrix}$$

Transforming the fuzzy numbers from this matrix using the G-index (10) and $\alpha = 0.5$ one has:

$$R^{tr}=\begin{bmatrix} 0.5 & 0.4924 & 0.5355 & 0.7537 \\ 0.6041 & 0.5 & 0.5998 & 0.8096 \\ 0.5506 & 0.5055 & 0.5 & 0.7555 \\ 0.3453 & 0.2894 & 0.3426 & 0.5 \end{bmatrix}.$$

This preference relation is a fuzzy preorder (reflexive, Δ-max transitive), according to [30], where properties of the operator Weighted Mean are investigated. Then, the perfect antisymmetry relation R' of R^{tr} [26] is computed as

$R(a,b) \geq R(b,a) \rightarrow R'(a,b) = R(a,b) \vee R'(b,a) = 0$ and therefore

$$R'=\begin{array}{c} \\ a_2 \\ a_3 \\ a_1 \\ a_4 \end{array}\begin{array}{c} \begin{array}{cccc} a_2 & a_3 & a_1 & a_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.5998 & 0.6041 & 0.8096 \\ 0 & 0.5 & 0.5506 & 0.7555 \\ 0 & 0 & 0.5 & 0.7537 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \end{array}.$$

The relation R' is a fuzzy partial ordering according to definition in [37] and it is obvious that this relation is a fuzzy linear ordering [15], i.e.

$$a_2 \xrightarrow{0.5998} a_3 \xrightarrow{0.5506} a_1 \xrightarrow{0.7537} a_4 .$$

5.2 Aggregation with crisp weighted coefficients

The transformed weighted coefficients of the fuzzy numbers (5) are (see Example 3.1): $F(\widetilde{w}_1) = 0.585$, $F(\widetilde{w}_2) = 0.415$. The membership degrees of the aggregated matrix R according to the method presented in section 4.1 are (see Example 4.1):

$$R=\begin{bmatrix} 0.5 & 0.5245 & 0.4925 & 0.6803 \\ 0.5537 & 0.5 & 0.5462 & 0.7340 \\ 0.5075 & 0.4538 & 0.5 & 0.6878 \\ 0.3197 & 0.2660 & 0.3122 & 0.5 \end{bmatrix}.$$

According to [30] for the properties of the operator Weighted Mean, it follows that this preference relation is a fuzzy preorder. Therefore the corresponding perfect antisymmetry relation R' and the linear fuzzy ordering of the alternatives from the set A are:

$$R' = \begin{matrix} & a_2 & a_3 & a_1 & a_4 \\ a_2 & 0.5 & 0.5462 & 0.5537 & 0.734 \\ a_3 & 0 & 0.5 & 0.5075 & 0.6878 \\ a_1 & 0 & 0 & 0.5 & 0.6803 \\ a_4 & 0 & 0 & 0 & 0.5 \end{matrix}, \qquad a_2 \xrightarrow{0.5462} a_3 \xrightarrow{0.5075} a_1 \xrightarrow{0.6803} a_4.$$

5.3.Aggregation by fuzzy relation between the criteria importance

The fuzzy relation between the criteria importance (5) from Example 3.2 is $W = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}$.

Therefore, the aggregated membership degrees of the relation R_{21} according to (11) are:

$$r_{12}(a,b) = \begin{cases} 0.5 & \text{if } a = b \\ S(T(1,\mu_1(a,b)),T(0,\mu_2(a,b))) = S(\mu_1(a,b),0) = \mu_1(a,b) & \text{if } a \neq b \end{cases},$$

i.e. $R_{12} = R_1$. But R_1 and R_2 are max-min transitive and W is additively reciprocal, therefore R_{21} is Δ-max transitive (see Proposition 4.2.1). The perfect antisymmetry relation R' of R_1 is:

$$R' = \begin{matrix} & a_1 & a_2 & a_3 & a_4 \\ a_2 & 0.5 & 0.55 & 0.7 & 0.95 \\ a_3 & 0 & 0.5 & 0.65 & 0.9 \\ a_1 & 0 & 0 & 0.5 & 0.75 \\ a_4 & 0 & 0 & 0 & 0.5 \end{matrix}.$$

As it is seen, the corresponding fuzzy linear ordering is: $a_1 \xrightarrow{0.55} a_2 \xrightarrow{0.65} a_3 \xrightarrow{0.75} a_4$.

5.4. Aggregation with composition

From $W = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}$ and (13) one has:

$$r_{ij} = \begin{cases} 0.5 & \text{if } a_i = a_j \\ S(T(1,z_{ij}^1),T(0,z_{ij}^2)) = S(z_{ij}^1,0) = z_{ij}^1 & \text{if } a_i \neq a_j \end{cases},$$

i.e. $R = Z^1 = X \circ Y$. According to (12) the max-min composition of the matrices (6) is:

$$Z^1 = X \circ Y = \begin{bmatrix} 0.665 & 0.475 & 0.380 & 0.475 \\ 0.63 & 0.45 & 0.36 & 0.45 \\ 0.525 & 0.375 & 0.3 & 0.375 \\ 0.35 & 0.25 & 0.2 & 0.25 \end{bmatrix}.$$

This composition is Δ-max transitive from Proposition 4.2.2. Then, the perfect antisymmetry relation R' and the linear fuzzy ordering of the alternatives from the set A are:

$$R'=\begin{array}{c|cccc} & a_3 & a_2 & a_1 & a_4 \\ \hline a_3 & 0.5 & 0.375 & 0.525 & 0.375 \\ a_2 & 0 & 0.5 & 0.63 & 0.45 \\ a_1 & 0 & 0 & 0.5 & 0.475 \\ a_4 & 0 & 0 & 0 & 0.5 \end{array}, \qquad a_3 \xrightarrow{0.375} a_2 \xrightarrow{0.63} a_1 \xrightarrow{0.475} a_4.$$

5.5. Comparison between orderings

The comparison of the orderings obtained by the presented above four methods show that the alternatives a_4 and a_1 are the "worst" ones. The orderings from examples presented in 5.1 and 5.2 are the same, but the method for obtaining the second ordering is simpler from a computational point of view. The ordering from the example given in section 5.3 is a quite different one. This is due to matrix W. The aggregated relation coincides with R_1 and therefore R_2 has no influence on R_{21}. This fact is compensated in the aggregated relation computed by the method using the composition between two fuzzy relations.

6. Concluding remarks

Weighted aggregations are important in decision making problems where one has multiple criteria to consider and where the outcome is to be judged in terms of criteria which are not equally important for the decision maker. It is more realistic to use fuzzy relations because they appear as a more convenient and adequate form for representing the relationship between alternatives then crisp relations. The fuzzy relations may model situations, whenever interactions between the alternatives are not exactly determined. Besides that, they reflect the interests of the experts or the decision-maker. The fuzzy numbers are a generalization of the real numbers (crisp ones) and they are more convenient for estimation of the criteria importance then the crisp weights.

Aggregation of fuzzy relations on the alternatives with the help of aggregation operators is the basic concept in the suggested above models. These models depend on the transformations of the fuzzy numbers into crisp weights or a fuzzy relation.

The aggregation operators with weighted coefficients are used when the weights are crisp numbers. Weighted transformations of aggregation operators for uniting fuzzy relations are used when the weights are not present in the mathematical formula of these operators. The proved connections between the properties of the individual fuzzy relations and the ones of the aggregated relations assist for solving the decision making problems.

A combination of t-norm and t-conorm is studied for obtaining a new fuzzy preference relation when fuzzy relation between the criteria importance is used. This relation connects the individual fuzzy preference relations with the relation between the fuzzy criteria importance. It is proved, that this relation preserves the transitivity property under the defined conditions.

Composition of two fuzzy preference relations in an aggregation procedure is investigated as well. It is proved, that the composition is max-Δ transitive, if both relations are additively or max-max transitive. It points out that the idea to use the composition to aggregate relations has a practical application. The newly fuzzy preference relation is used after that. It is proved that the aggregated fuzzy preference relation preserves the max-Δ transitivity of the composition under the defined conditions.

References

[1] S.Alonso, F.Chiclana, F.Herrera, E.Herrera-Viedma, J.Alcala-Fdez, and C.Porcel, A consistency based procedure to estimate missing pairwise preference values, in: Int. J. of Intelligent Systems (2005), http://decsai.ugr.es/~viedma/public.html

[2] G.Bortolan and R. Degani, A review of some methods for ranking fuzzy subsets, Fuzzy Sets and Systems 15 (1985) 1-19.

[3] W.Chang, Ranking of fuzzy utilities with triangular membership function, in: Proc. Int. Conf. on Policy Anal. And Inf. Systems (1981) 263-276.

[4] Ch.B. Chen and, C.M. Klein, Fuzzy ranking methods for multi-attribute decision making, IEEE Conf. on SMC, San Antonio, USA (1994) 475-480.

[5] F.Chiclana, F.Herrera and E.Herrera-Viedma, The ordered weighted geometric operator: Properties and applications in: Proc. of 7th IPMU'2000, Int. Conf. on Inf. Proc. and Manag. of Univ. in Knowledge-Bases Systems, IPMU'2000, vol. II (DECSAI University of Granada, 2000) 985-991.

[6] F.Chiclana, F.Herrera, E.Herrera-Viedma and L.Martinez, A Note on the Reciprocity in the Aggregation of Fuzzy Preference Relations Using OWA Operators, Fuzzy Sets and Systems 137 (2003) 71-83.

[7] F.Chiclana, F.Herrera and E.Herrera-Viedma, Rationality of Induced Ordered Weighted Operators Based on the Reliability of the Source of Information in Group Decision – making, Kybernetika 40 (2004) 121-142.

[8] G.Bortolan and R. Dagani, A review of some methods for ranking fuzzy subsets, Fuzzy Sets and Systems 15 (1985) 1-19.

[9] M. Delgado, J.L. Verdegay and M.A. Vila, A procedure for ranking fuzzy numbers using fuzzy relations, Fuzzy Sets and Systems 26 (1988) 49-62.

[10] M.Detyniecki, Mathematical Aggregation Operators and Their Application to Video Querying, in: Thesis for the degree Docteur de l'Universite Paris VI (2001), www.lip6.fr/reports/lip6.2001.html

[11] M. Detyniecki and R.R.Yager, Ranking fuzzy numbers using α-weighted valuations, Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems 8, No 5 (2001) 573-592.

[12] O.P. Dias, Ranking alternatives using fuzzy numbers: A computational approach, Fuzzy Sets and Systems 56 (1993) 247-252.

[13] J.Drewniak and U. Dudziak, Safe transformations of fuzzy relations, in: Proc. of EUROFUSE 2004, The 8-th Meeting of EURO Working Group on Fuzzy Sets, Workshop on Data and Knowledge Engineering, Sept. 22-25, 2004, Warszawa, Poland (2004) 195-203.

[14] D.Dubois and H. Prade, Operations on fuzzy numbers, Int. J. System Sci. 9 (1978) 613-626.

[15] D.Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications (New York, Academic Press, 1980)

[16] D.Dubois and H. Prade, Ranking of fuzzy numbers in the setting of possibility theory, Inf. Sci. 30 (1983) 183-224.

[17] D.Dubois and H. Prade, On the use of aggregation operations in information fusion processes, Fuzzy Sets and Systems 142 (2004) 143-161.

[18] P.Ekel W. Pedrycz and R. Schinzinger, A general approach to solving a wide class of fuzzy optimization problems, Fuzzy Sets and Systems 97 (1998) 49-66.

[19] M.Grabisch, S. Orlowski and R. Yager, Fuzzy aggregation of numerical preferences, in: Slowinski R., ed., Fuzzy Sets in Decision Analysis Operations Research and Statistics ,The Handbooks of Fuzzy Sets Series (Kluwer, Boston, USA, 1998) 31-68.

[20] E.Herrera-Viedma, Herrera, F. Chiclana and M. Luque, Some Issues on Consistency of fuzzy preference relations, European Journal of Operational Research 154 (2004) 98-109.

[21] J.Jacas and J.Recasens, Aggregation of T-transitive Relations, in: Int. J. of Intelligent Systems 18(12) (2003) 1193-1214.

[22] K.Kim and K.S. Park, Ranking fuzzy numbers with index of optimism, Fuzzy Sets and Systems 35 (1993) 143-150.

[23] W.Kolodziejczyk, Orlovsky's concept of decision-making with fuzzy preference relation further results, Fuzzy Sets and Systems 19 (1986) 11-20.

[24] S.Mabuchi, An approach to the comparison of fuzzy subsets with α -cut dependent index, IEEE Trans. on Systems, Man and Cybernetics 18(2) (1988) 264-272.

[25] R.Mesiar and S.Saminger, Domination of ordered weighted averaging operators over t-norms, Soft Computing 8 (2004) 562-570.

[26] K.Nakamura, Preference relations on a set of fuzzy utilities as a basis for decision making, Fuzzy Sets and Systems 20 (1986) 147-162.

[27] H.Nguyen and E.Walker, Fuzzy Logic (Chapman & Hall/CRC, 2000).

[28] V.Peneva and I.Popchev, Comparison of cluster from fuzzy numbers, Fuzzy Sets and Systems 97(1) (1998) 75-81.

[29] V.Peneva and I. Popchev, Aggregation of fuzzy numbers in a decision making situation, Cybernetics and Systems 32(8) (2001) 821-835.

[30] V.Peneva and I.Popchev, Properties of the aggregation operators related with fuzzy relations Fuzzy Sets and Systems 139 (2003) 615-633.

[31] V.Peneva and I.Popchev, Transformations by parameterized t-norms preserving the properties of fuzzy relations, Compt. Rend. Acad. Bulg. Sci. 57(10) (2004) 9-18.

[32] V.Peneva and I.Popchev, Weighted transformation in aggregation operators uniting fuzzy relations, Advanced Studies in Contemporary Mathematics 10 (1) (2005) 25-44.

[33] V.Peneva and I.Popchev, Aggregation of fuzzy preference relations with different importance, Compt. Rend. Acad. Bulg. Sci. 58(5) (2005) 499-506.

[34] V.Peneva and I.Popchev, Aggregation of fuzzy preference relations to multicriteria decision making, Fuzzy Optimization and Decision Making (to appear).

[35] M.Roubens, Some properties of choice functions based on valued binary relations, European J. Oper. Res. 40 (1989) 309-321

[36] T.Tanino, Fuzzy preference relations in group decision making, in: J. Kacprzyk, M. Roubens, eds., Non-Conventional Preference Relations in Decision Making (Springer-Verlag, Berlin 1988).

[37] P.Venugopalan, Fuzzy ordered sets, Fuzzy Sets and Systems 46 (1992) 221-226.

[38] Z.Xu, and Q.Da, An approach to improving consistency of fuzzy preference matrix, Fuzzy Optimization and Decision Making 2 (2003) 3-12.

[39] R.Yager, A procedure for ordering fuzzy subsets on unit interval, Inf. Sci. 24 (1981) 143-161.

[40] Y.Yuan, Criteria for evaluating fuzzy ranking methods, Fuzzy Sets and Systems 44 (1991) 139-157.

[41] L.A. Zadeh, A Similarity Relations and Fuzzy Orderings, Infom. Sci. 3 (1971) 177-200.